HOMOLOGY CYCLE BASES FROM ACYCLIC MATCHINGS

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Kyoto Workshop, January 2019
What is ... 

**Applied Topology?**

- studying global features of shapes
- applications in other branches of mathematics
- applications in computation and in other sciences
Combinatorial cell complexes

locally = simple cell attachments
for example - simplicial or prodsimplicial complexes;

globally = cells are indexed by combinatorial objects
such as graphs, partitions, permutations, various combinations and enrichments of these;

taking boundary = combinatorial rule for the indexing objects
for example - removing vertices in graphs, merging blocks in partitions, relabeling, etc.

Combinatorial means
using matchings, orderings, labelings, et cetera, to simplify or to completely eliminate
algebraic computations and topological deformations.
A very useful tool is provided by the notion of simplicial collapses.

**Definition.**

An *elementary simplicial collapse* in a simplicial complex $K$ is the removal of a pair of simplices $(\sigma, \tau)$, such that

- $\tau$ is a maximal simplex;
- $\dim \sigma = \dim \tau - 1$;
- $\tau$ is the only maximal simplex containing $\sigma$.

A simplicial complex $K$ is called **collapsible** if there exists a sequence of elementary simplicial collapses reducing $K$ to a vertex.
The idea of Discrete Morse Theory.

The homotopy type is clear if we can remove some top-dimensional simplices and obtain a collapsible complex.

More generally, we may want to make **internal collapses** and trace how the remaining **critical** simplices are glued on afterwards.
Definition.
Assume we are given a simplicial complex $K$, and a partial matching $\mu : W \to W$ on the set of simplices of $K$.
This matching is called **acyclic**, if there does not exist a cycle of the following form:

$$b_1 \succ \mu(b_1) < b_2 \succ \mu(b_2) < \cdots < b_n \succ \mu(b_n) < b_1,$$

with $n \geq 2$, and all $b_i \in W$ being distinct.

Theorem.
Assume $K$ is an abstract simplicial complex, and assume $K'$ is a simplicial subcomplex of $K$. The following statements are equivalent:

1. there exists a sequence of elementary collapses leading from $K$ to $K'$;
2. there exists an acyclic matching on the set of the simplices of $K$ which are not contained in $K'$. 
Theorem.
Assume $K$ is a simplicial complex, $M$ is some set of simplices of $K$, and $\mu : M \to M$ is an acyclic matching. Then there exists a CW complex $X$ such that

1. for each dimension $d$, the number of $d$-cells in $X$ is equal to the number of $d$-simplices of $K$, which do not belong to $M$.
2. we have a homotopy equivalence $K \simeq X$.

The simplices of $K$ which do not belong to $M$ are called critical with respect to $\mu$.

This theorem is a very important result since it allows us to replace $K$ with a potentially much smaller CW complex.
Definition.

Assume we are given two arbitrary posets $P$ and $Q$, and a poset map $\varphi : P \rightarrow Q$. The map $\varphi$ is called a **poset map with small fibers**, if for any $q \in Q$, one of the following three statements is true:

- the fiber $\varphi^{-1}(q)$ is empty;
- the fiber $\varphi^{-1}(q)$ consists of a single element;
- the fiber $\varphi^{-1}(q)$ consists of two comparable elements.

In particular, we have a partial matching $M(\varphi)$.

**Theorem: Acyclic matchings via poset maps with small fibers.**

The following two statements describe the close relation between poset maps with small fibers and acyclic matchings.

1. For any poset map with small fibers $\varphi : P \rightarrow Q$, the partial matching $M(\varphi)$ is acyclic.
2. Any acyclic matching on $P$ can be represented as $M(\varphi)$ for some poset map with small fibers $\varphi$. 
More in the upcoming book on *Discrete Morse theory.*
Definition.

Assume we are given an abstract simplicial complex $K$, and an acyclic matching $\mu$. Pick $0 \leq d \leq \dim K$. The oriented graph $G_d(\mu)$ is defined as follows:

- the vertices of $G_d(\mu)$ are indexed by the $d$-dimensional simplices of $K$;
- the edges of $G_d(\mu)$ are given by the rule: $(\alpha, \beta)$ is an edge of $G_d(\mu)$ if and only if $\mu_-(\beta)$ is defined, and $\alpha \succ \mu_-(\beta)$.

The rule defining the edges of the oriented graph $G_d(\mu)$. 
The recursive rule for $\varphi$

$$\varphi(\alpha) := \alpha + \sum_{i=1}^{m} \varphi(\beta_i)$$

**Theorem.** Values of $\varphi$ on critical simplices give a homology basis.
A source of combinatorial complexes is provided by complexes associated to monotone graph properties.

**Definition.**
Assume \( n \) is a natural number. A **graph property** \( \Gamma \) is a collection of isomorphism classes of graphs on the vertex set \( \{1, \ldots, n\} \).

A graph property is called **monotone** if: whenever a graph \( G \) has property \( \Gamma \), any graph obtained from \( G \) by the deletion of some of the edges will also have the property \( \Gamma \).

Given a non-trivial monotone graph property \( \Gamma \), we define the simplicial complex \( X(\Gamma) \) as follows:

- the vertices of \( X(\Gamma) \) are all potential edges, that is, all pairs \((i, j), 1 \leq i < j \leq n\);

- the set of vertices forms a simplex of \( X(\Gamma) \) if the corresponding graph has the property \( \Gamma \).

For example, one can consider the **simplicial complex of all disconnected graphs** (but not of all connected graphs!).
Cycles for the complex of disconnected graphs.

Standard acyclic labeling is given iteratively by first trying $\sigma \text{ XOR } (1, 2)$, then $\sigma \text{ XOR } (i, 3)$ etc.

$\varphi(\sigma) = \partial(T_\sigma)$, where $T_\sigma$ is a recursive tree.
**Definition.**

Given a positive integer $n$, the **partition lattice** $\Pi_n$ consists of all proper partitions of the set $\{1, \ldots, n\}$, ordered by refinement.

We see that:

- the vertices of the simplicial complex $\Delta(\Pi_n)$ are indexed by all proper partitions of the set $\{1, \ldots, n\}$,
- the simplices of $\Delta(\Pi_n)$ are all refinement chains of partitions.

In particular, the simplicial complex $\Delta(\Pi_n)$ is **pure**, meaning all its top-dimensional simplices have the same dimension, and it has dimension $n - 3$. 
For example, for $n = 4$ we get the following simplicial complex.
**Theorem.**

The simplicial complex $\Delta(\Pi_n)$ is homotopy equivalent to a wedge of $(n - 1)!$ spheres of dimension $n - 3$.

So, for example,

- $\Delta(\Pi_4)$ is homotopy equivalent to a wedge of 6 spheres of dimension 1,
- $\Delta(\Pi_5)$ is homotopy equivalent to a wedge of 24 spheres of dimension 2, etc.
Cycles for the partition lattice.

Any chain $\sigma$ in $\Delta(\Pi_n)$ can be written as

$$(\alpha_1 n) < \cdots < (\alpha_1 \ldots \alpha_k n) < \pi < \ldots$$

where $\pi$ is not of the form $(\alpha_1 \ldots \alpha_k \alpha_{k+1} n)$.

Set $B := \{\alpha_1 \ldots \alpha_k n\}$. The pivot is the partition obtained by chopping off $B$ in the first possible partition in $\sigma$.

Set $\mu(\sigma) := \mu(\sigma)$ XOR pivot.

Our technique finds the classes associated to imbedded Boolean algebras.
Graph colorings
Graph colorings
**Graph colorings**

**Chromatic number** of a graph $G$, denoted $\chi(G)$, is the smallest number of colors needed to color the vertices of $G$ so that no two adjacent vertices share the same color.

Calculating the chromatic number of a graph is an NP-complete problem.
The topology of graph colorings

Ansatz

\[
\begin{align*}
\text{graph } G & \quad \longrightarrow \quad \text{topological space } X(G) \\
\downarrow & \\
\text{combinatorial properties of } G & \quad \longleftarrow \quad \text{topological properties of } X(G)
\end{align*}
\]
The topology of graph colorings

**Definition.**
A **graph homomorphism** between graphs $T$ and $G$ is a map $\varphi : V(T) \to V(G)$, such that for every edge $(x, y)$ in $T$ the pair $(\varphi(x), \varphi(y))$ is an edge in $G$.

**Observation:** $G$ is $n$-colorable $\iff \exists \varphi : G \to K_n$

The composition of two homomorphisms $\varphi_1 : G_1 \to G_2$ and $\varphi_2 : G_2 \to G_3$ is again a homomorphism $\varphi_2 \circ \varphi_1 : G_1 \to G_3$.

We obtain the category **Graphs** with graphs as objects and graph homomorphisms as morphisms.
The topology of graph colorings

A cell in $\text{Hom}(T, K_n)$ is an assignment of subsets of $[n]$ to vertices of $T$, such that an arbitrary choice of one color per list yields an admissible coloring of $T$.

Now replace $K_n$ with an arbitrary graph $G$.

- The vertices of $G$ are the colors.
- Homomorphisms $T \to G$ replace the valid colorings.

Example.

\[
\text{Hom}(\begin{array}{cc}
\text{\includegraphics[width=0.1\textwidth]{triangle.png}}
\end{array}, \begin{array}{cc}
\text{\includegraphics[width=0.1\textwidth]{triangle.png}}
\end{array}) = \begin{array}{cc}
\text{\includegraphics[width=0.3\textwidth]{triangles.png}}
\end{array}
\]
$\text{Hom}(C_6, K_3)$
Consider the special case $\text{Hom}(K_m, K_n)$, where $K_m$ and $K_n$ are complete graphs on $m$, resp. $n$, vertices.

In that case the cells are indexed by all $m$-tuples $(A_1, \ldots, A_m)$, such that $A_i$'s are non-empty disjoint subsets of $[n]$.

In particular, the vertices of $\text{Hom}(K_m, K_n)$ are simply all $m$-tuples $(v_1, \ldots, v_m)$, such that $v_i \in [n]$, and $v_i \neq v_j$, while for a maximal-dimensional cell $(A_1, \ldots, A_m)$, we have $A_1 \cup \cdots \cup A_m = [n]$. The combinatorial rule for the boundary operation is the removal of elements from $A_i$'s.
Acyclic matching for $\text{Hom}(K_m, K_n)$ (Idea):

start scanning from the right and try to insert the maximal missing element so that it comes out on top of this set.
A cycle in \( \text{Hom}(K_3, K_5) \).
Algorithm 1  The algorithm computing explicit homology cycles

for all $d$-cells $v$ do
  set $out(v) := 0$ \{Initialize the outdegrees\}
  for all $d$-cells $w$ do
    set $G(v, w) := \text{false}$ \{Initialize the graph\}
  end for
end for

for all $d$-cells $v$ do
  for all $(d-1)$-cells $u$ in $\partial v$ do
    if $u \in N_{d-1}$ then
      set $G(v, \mu(u)) := \text{true}$ \{Create an edge\}
      set $out(v) := out(v) + 1$ \{Increase the outdegree\}
    end if
  end for
end for

for all $d$-cells $v$ do
  set $\varphi(v) := v$ \{Initialize the $\varphi$ function\}
end for

let $S$ be the set of all $v$, such that $out(v) = 0$ \{Put all the sinks in $S$\}

repeat
  pick $v \in S$
  for vertices $w$, such that $G(w, v)$ do
    set $out(w) := out(w) - 1$ \{The edge $(w, v)$ is spent\}
    set $\varphi(w) := \varphi(w) + \varphi(v)$ \{Update the value of $\varphi$ on $w$\}
    if $out(w) = 0$ then
      add $w$ to the set $S$ \{If $w$ is a new sink, then add it to the list\}
    end if
  end for
remove $v$ from $S$ \{The vertex $v$ is spent\}
until the set $S$ is empty