Fault-tolerant protocols and trace spaces

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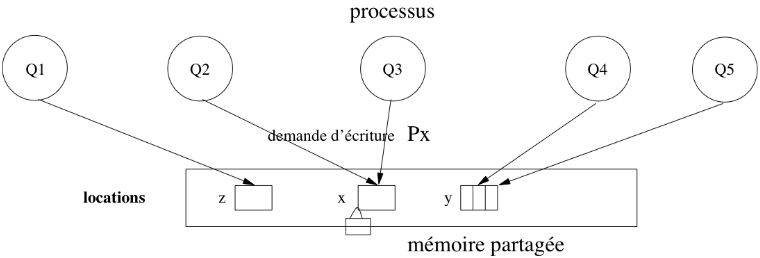
Contents of the talk

- Some directed algebraic topology, in the shared memory, semaphore case: trace spaces
- A quick recap on fault-tolerants protocols for distributed systems (here, immediate snapshot and layered executions protocols à la Maurice Herlihy et al.)
- Links between the two approaches and future work

(ongoing work, with lots of inputs from Samuel Mimram, Emmanuel Haucourt, Christine Tasson, Lisbeth Fajstrup, Martin Raussen)

Context of this talk

- We consider in this talk *concurrent programs interacting through shared memory* (as an example)



- Synchronisation:
  - through semaphores (P for locking, V for unlocking), binary or "counting"
  - Or synchronisation through *scan/update*

Quick history

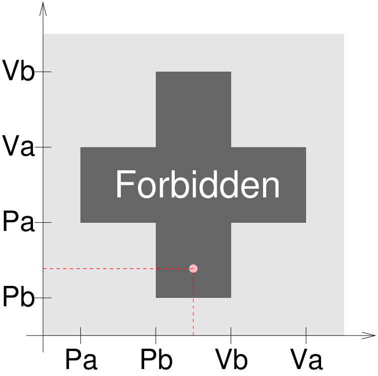
- "Progress graph" model of E. W. Dijkstra (1968)
- Applications to deadlock finding and correctness of distributed databases (serializability), Yannakakis, Lipsky, Papadimitriou etc. (1979-1985), Gunawardena (2 phase-locking protocol, 1994) etc.
- "Higher-dimensional automata" as a model for concurrency, Pratt/Van Glabbeek 1991, Goubault 1992, Raussen, Fajstrup, Grandis, Gaucher, etc., applications to static analysis of concurrent systems (state-space reduction)

(and many influences of other geometrical aspects of computer science, "Squier's theorem" 1985, Univalent Foundations of Voevodsky/Awodey 2009 etc.)

Link with (the classical) algebraic topological approach in DC?

"progress graphs" E.W.Dijkstra'68

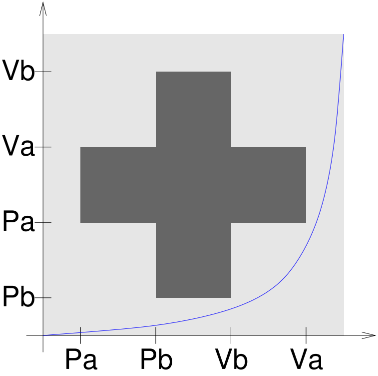
T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



"Continuous model": x_i = local time; dark grey region=forbidden!

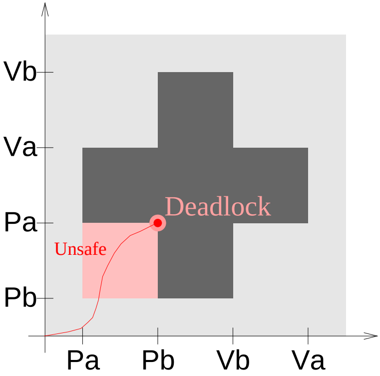
see *Algebraic Topology and Concurrency* MFPS 1998/TCS 2006, L. Fajstrup, E. Goubault, M. Raussen

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb

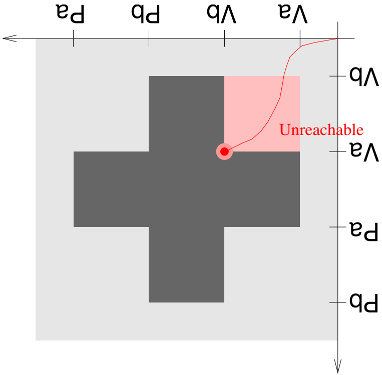


Traces are continuous paths increasing in each coordinate: dipaths.

Deadlocks



Unreachables



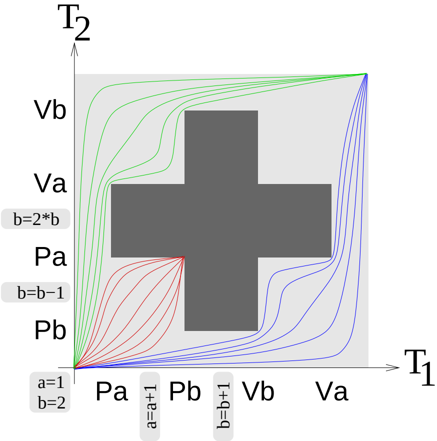
Classes of equivalent dipaths

up to dihomotopy

T1 gets a and b before T2 => a=2 and b=4

T2 gets b and a before T1 => a=2 and b=3

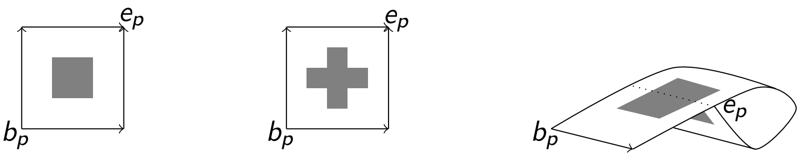
Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1



Examples of geometric semantics

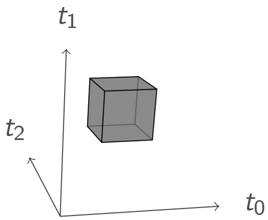
To each program p we associate a directed space of some sort (d-space, stream etc.):

$P_a.V_a|P_a.V_a$ $\quad$ $P_a.P_b.V_b.V_a|P_b.P_a.V_a.V_b$ $\quad$ $P_a.(V_a.P_a)^*|P_a.V_a$

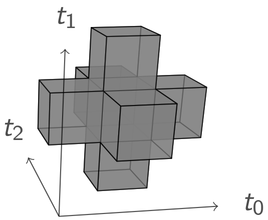


Examples of PV semantics

$P_a.V_a|P_a.V_a|P_a.V_a$
$(\kappa_a = 2)$



$P_a.V_a|P_a.V_a|P_a.V_a$
$(\kappa_a = 1)$



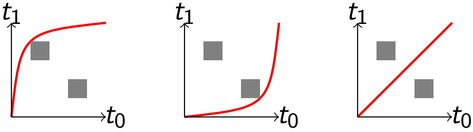
Basic definitions in directed algebraic topology

- Let X be a stream/d-space etc. (here we only consider a po-space, i.e. a topological space X together with a partial order $\leqslant \subseteq X \times X$, closed in the product topology)
- $p : I \to X$ a continuous and increasing path from po-space $I = ([0,1], \leqslant)$ (standard order) to X is a *directed path*
- Define the path space $P(X)(a,b) = \{p : I \to X$ mod $p(0) = a, p(1) = b, p$ is a directed path$\}$
- A *dihomotopy* on $P(X)(a,b)$ is a continuous map $H : I \times I \to X$ such that $H_t \in P(X)(a,b)$ for all $t \in I$.

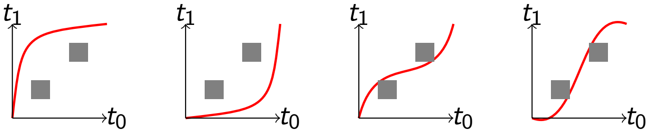
Fact

Schedules are dihomotopy classes of dipaths

Dihomotopy equivalence is finer than homotopy equivalence



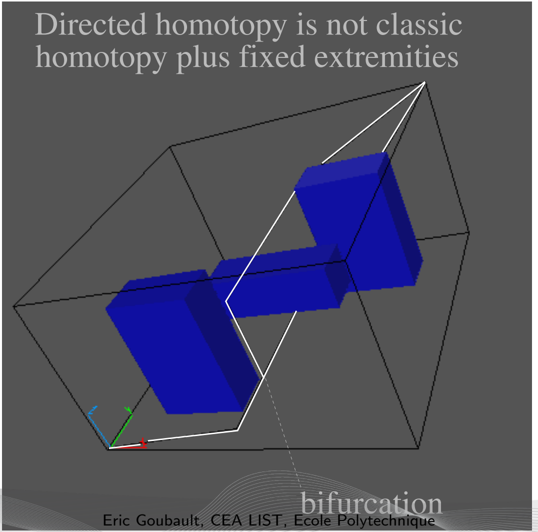
($Pa.Va.Pb.Vb$ | $Pb.Vb.Pa.Va$, 3 maximal schedules) Different from:



($Pa.Va.Pb.Vb$ | $Pa.Va.Pb.Vb$, 4 maximal schedules)

But, as topological spaces, they are homotopy equivalent!

$Pa.Pc.Va.Pb.Vc.Vb \mid Pa.Va.Pc.Vc.Pb.Vb \mid Pc.Vc\ (c:2)$



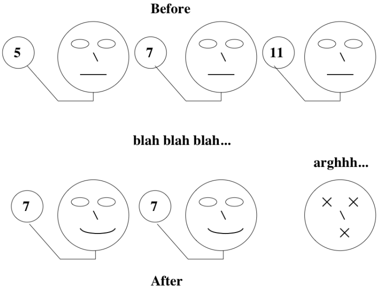
DC - case of the update-scan model

Update-scan model, very close to the PV model:

- Each process P_i has a distinguished local variable x_i
- It can *update* the value of its "mirror" in global memory X_i; $(X_i,\ i = 0, \ldots, n-1)$ forms a partition of global memory
- It can *scan* all of the global memory into its local memory
- It can perform local computations...

Processes are supposed to do (update; computation; scan)* in parallel

Can we implement a function...given an "architecture" (faults? shared memory / message passing, synchronous / semi-synchronous / asynchronous etc.)?

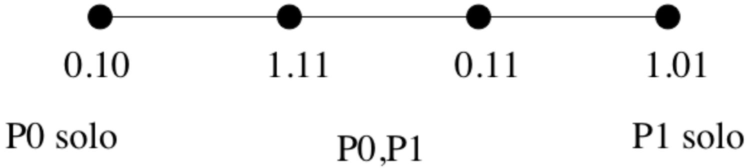


Protocol complex

Each protocol on some architecture defines:

- a simplicial set (for all rounds r):
  - vertices: sequence of "values" scanned at a given round r
  - simplices: compound states at round r
- This is an operator on an input simplex
- A choice of model of computation entails some geometrical properties of the protocol complex

One-round protocol simplicial set (2D)



- First digit is the process number (identifying the local state)
- After the dot, for each round, we get a string of n bits, where n is the number of processes involved (here just one round, and $n = 2$)

One-round protocol simplicial set (3D)

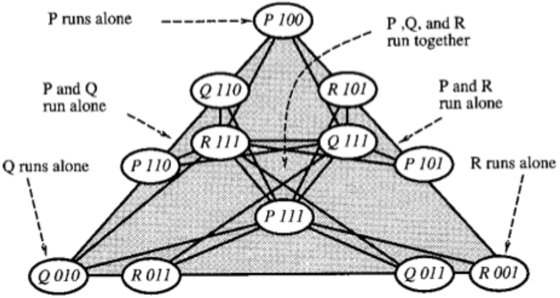


FIG. 25. A one-round protocol complex.

How can we find such pictures?

How can we make the link between the two approaches?

- But where does the protocol complex comes from? The different local states should come from different schedules of execution

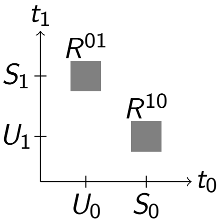
  The higher dimensional simplexes in the protocol complexes will correspond to distinct schedules (i.e. paths mod dihomotopy classes)

- To be computed from the (geometric) semantics of some "generic" scan/update program

How can we generalize this to more intricate distributed models, than scan/update?

Examples of *scan*/*update* semantics

Only "obstructions" are between *scan* and *update*:



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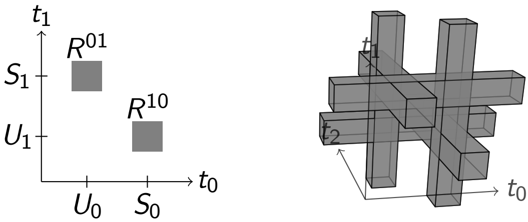


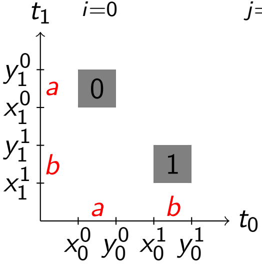
In dimension n, the forbidden region consists of n crosses with $n-1$ orthogonal branches.

For this talk - just po-spaces

Suppose given a program with n *threads* $p = p_0|p_1|\ldots|p_{n-1}$
Under mild assumptions, the geometric semantics is of the form

$$G_p = \vec{I}^n \setminus \bigcup_{i=0}^{l-1} R^i;\ R^i = \prod_{j=0}^{n-1}]x_j^i, y_j^i[$$

Example:



Formally

- Let X be a stream/d-space etc.
- Define the *trace space* $T(X)(a,b)$ to be the path space between a and b modulo continuous and increasing reparametrizations

- We wish to study the homotopy type of $T(X)(a,b)$
- There is a homotopy equivalence between $T(X)(a,b)$ and a certain prodsimplicial complex (Martin Raussen), which can be calculated combinatorially, on our simple semantics...

Determining traces can be intricate!

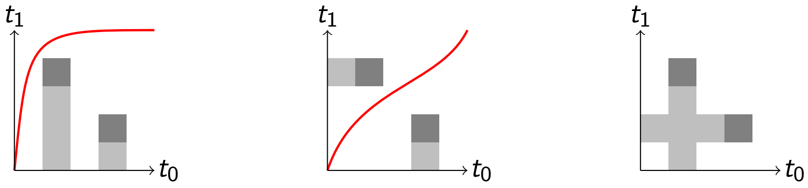
Px.Py.Pz.Vx.Pw.Vz.Vy.Vw | Pu.Pv.Px.Vu.Pz.Vv.Vx.Vz Py.Pw.Vy.Pu.Vw.Pv.Vu.Vv

In fact...

- Binary semaphores are "easy" (trace spaces are discrete!)
- In general (with counting semaphores), recent result by Krzysztof Ziemański (unpublished, 2013):
  For each finite simplicial set S, there exists a finite PV-program P such that the trace space of P (from beginning to end) is homotopy equivalent to S
- So we may have the complexity of general homotopy types even with a simple computational model such as PV...

Determining trace spaces, combinatorially

The main idea is to extend the forbidden cubes downwards in various directions and look whether there is a path from b to e in the resulting space.

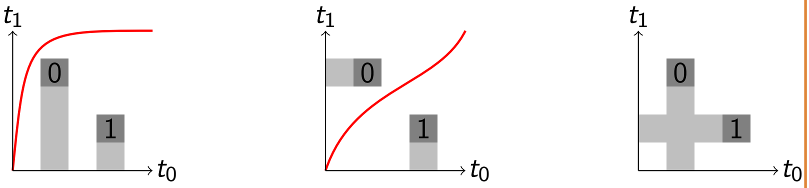


By combining those information, we will be able to compute traces modulo homotopy.

The directions in which to extend the holes will be coded by boolean matrices M.

$\mathcal{M}_{l,n}$: boolean matrices with l rows and n columns.

X_M: space obtained by *extending* for every (i,j) such that $M(i,j) = 1$ the forbidden cube i downwards in every direction other than j



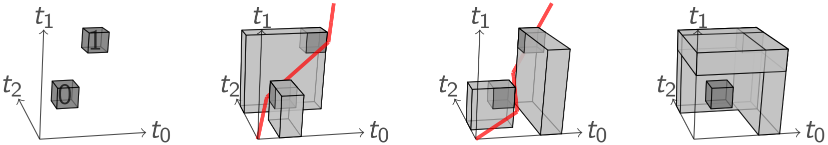
$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$ alive

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ alive

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ dead

The index poset, combinatorially

$$P_a.V_a.P_b.V_b \quad | \quad P_a.V_a.P_b.V_b \quad | \quad P_a.V_a.P_b.V_b$$



$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

alive *alive* *alive* *dead*

Alive and dead?

Important matrices are

- the **dead poset** $D(X) = \{M \in \mathcal{M}^C_{l,n} \;/\; \Psi(M) = 1\}$.
- the **index poset** $\mathcal{C}(X) = \{M \in \mathcal{M}^R_{l,n} \;/\; \Psi(M) = 0\}$ (the alive matrices).
- consider the entrywise ordering $(0 < 1)$ on matrices.

General results by Martin Raussen:

$D(X) \quad \rightsquigarrow \quad \mathcal{C}(X) \quad \rightsquigarrow$ homotopy classes of traces

(and even more, but let us just start with that!)

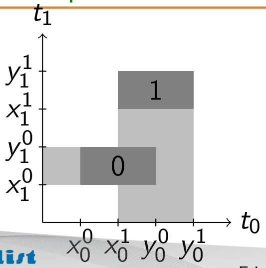
Proposition

A matrix $M \in \mathcal{M}^C_{l,n}$ is in $D(X)$ iff it satisfies

$$\forall (i,j) \in [0:l[\times[0:n[, \qquad M(i,j)=1 \quad \Rightarrow \quad x^i_j < \min_{i' \in R(M)} y^{i'}_j$$

where $R(M)$: indexes of non-null rows of M.

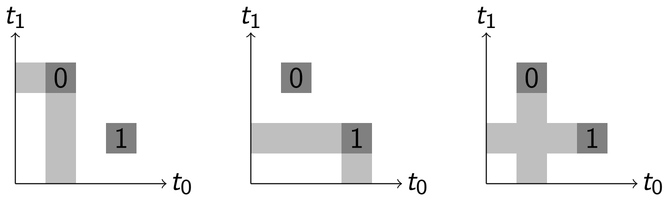
Example



$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \begin{array}{l} x^0_1 = 1 < 2 = \min(y^0_1, y^1_1) \\ x^1_0 = 2 < 3 = \min(y^0_0, y^1_0) \end{array}$$

Example, scan/update in dimension 2

3 dead matrices



$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Proposition

A matrix M is in $\mathcal{C}(X)$ iff for every $N \in D(X)$, $N \not\leq M$.

Remark

$N \not\leq M$: *there exists* (i,j) *s.t.* $N(i,j) = 1$ *and* $M(i,j) = 0$.

Remark

Since $\mathcal{C}(X)$ is downward closed it will be enough to compute the set $\mathcal{C}_{\max}(X)$ of maximal alive matrices.

Definition

Two matrices M and N are **connected** when $M \wedge N$ does not contain any null row. ($M \wedge N$: pointwise min of M and N)

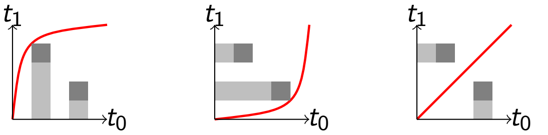
Proposition

The connected components of $\mathcal{C}(X)$ are in bijection with homotopy classes of traces $b \to e$ in X.

Scan/update in dimension 2 - 1 round

$$u.s \mid u.s$$

generates a trace space made of 3 distinct points:



$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Hypergraph transversal

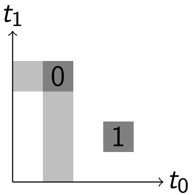
- An *hypergraph* $H = (V, E)$ consists of a set V of *vertices* and a set E of edges, where an *edge* is a subset of V
- A *transversal* T of H is a subset of V such that $T \cap e \neq \emptyset$ for every edge $e \in E$.

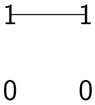
$D(X) \Rightarrow$ hypergraph H:

- vertices: $[0 : l[\times[0 : n[$
- hyperedges: $\{(i, j) \ / \ D(i, j) = 1\}$ (D is a matrix in $D(X)$)

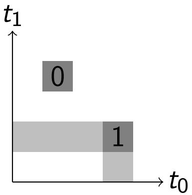
The sets $\{(i, j) \ / \ M(i, j) = 0\}$, where M is a maximal matrix of $\mathcal{C}(X)$, correspond to *minimal transversals* (wrt inclusion order) of H.

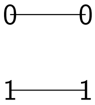
First dead matrix:



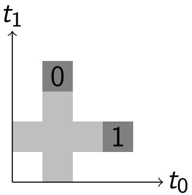


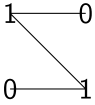
Second dead matrix:





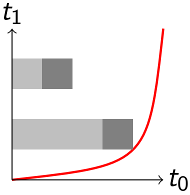
Third and last (minimal) dead matrix:

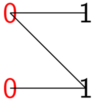




Some combinatorial considerations

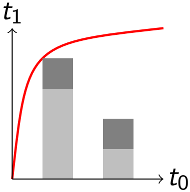
First (maximal) alive matrix:

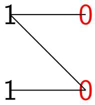




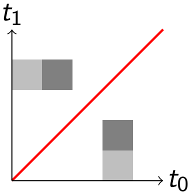
Some combinatorial considerations

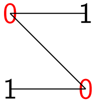
Second alive matrix:



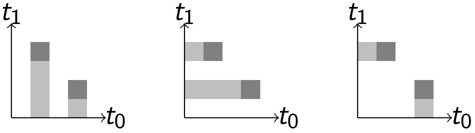


Third (and last) maximal alive matrix:





What is the *meaning* of traces?



$$\underbrace{\begin{pmatrix}1 & 0\\1 & 0\end{pmatrix}}_{M_1} \qquad \underbrace{\begin{pmatrix}0 & 1\\0 & 1\end{pmatrix}}_{M_2} \qquad \underbrace{\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}}_{M_3}$$

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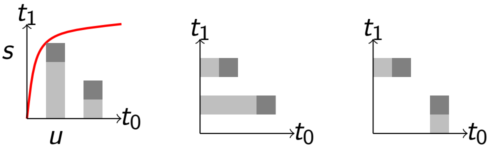

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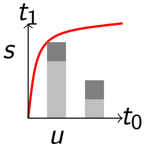
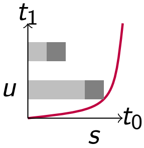
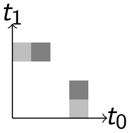
- M_1: P_1 does its scan before P_0 does its update
- M_1: P_1 *does not know* the current value of P_0 but P_0 does

What is the *meaning* of traces?



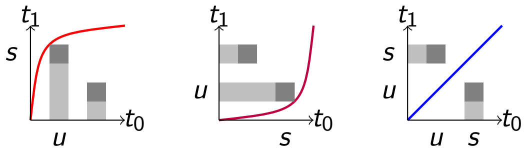




$$\underset{M_1}{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}} \qquad \underset{M_2}{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} \qquad \underset{M_3}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

- M_1: P_1 does its scan before P_0 does its update
- M_2: P_0 does its scan before P_1 does its update
- M_1: P_1 *does not know* the current value of P_0 but P_0 does
- M_2: P_0 *does not know* the current value of P_1 but P_1 does

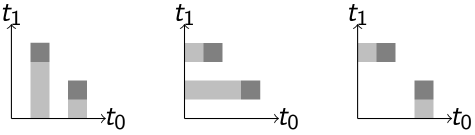
What is the *meaning* of traces?



$$M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

- M_1: P_1 does its scan before P_0 does its update
- M_2: P_0 does its scan before P_1 does its update
- M_3: P_0 and P_1 do update, then do there scan together
- M_1: P_1 *does not know* the current value of P_0 but P_0 does
- M_2: P_0 *does not know* the current value of P_1 but P_1 does
- M_3: P_0 and P_1 *know* their values

Link with the protocol complex



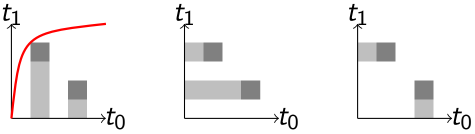
$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Protocol complex:

$$1.01 \longrightarrow 0.11 \longrightarrow 1.11 \longrightarrow 0.10$$

(1.01 (resp. 0.10) means P_1 (resp. P_0) knows only its own value; 1.11 (resp. 0.11) means P_1 (resp. P_0) knows all values)

Link with the protocol complex



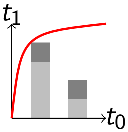
$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

M_1

Protocol complex:

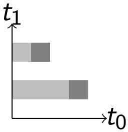
$$1.01 \xrightarrow{M_1} 0.11 \longrightarrow 1.11 \longrightarrow 0.10$$

Link with the protocol complex

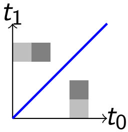


$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

M_1



$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

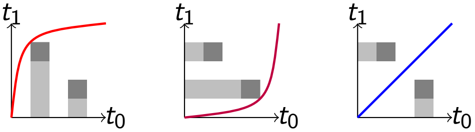
M_3

Protocol complex:

$$1.01 \xrightarrow{M_1} 0.11 \xrightarrow{M_3} 1.11 \longrightarrow 0.10$$

M_3 differs from M_1 by just a 1 (connected)

Link with the protocol complex



$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$M_1 \qquad M_2 \qquad M_3$$

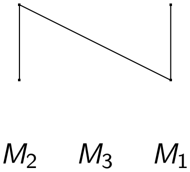
Protocol complex:

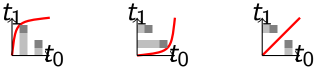
$$1.01 \xrightarrow{M_1} 0.11 \xrightarrow{M_3} 1.11 \xrightarrow{M_2} 0.10$$

M_3 differs from M_2 by just a 1 (connected)

This is actually the minimal transversal hypergraph!

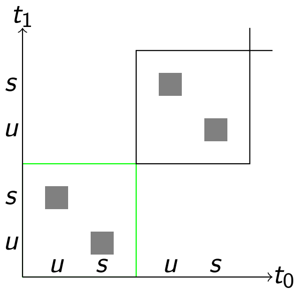
(vertices are indexes in the matrices, hitting sets are hyper-edges):





$$\underset{M_1}{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}} \quad \underset{M_2}{\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}} \quad \underset{M_3}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

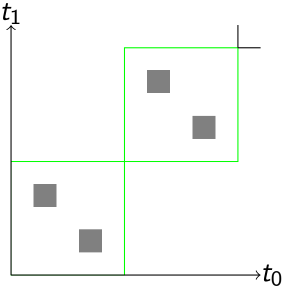
More rounds? clean-memory/layered immediate snapshot



Iterated subdivision (fractal) of the protocol complex (round 1):

$$1.01 \xrightarrow{M_1} 0.11 \xrightarrow{M_3} 1.11 \xrightarrow{M_2} 0.10$$

Clean-memory model



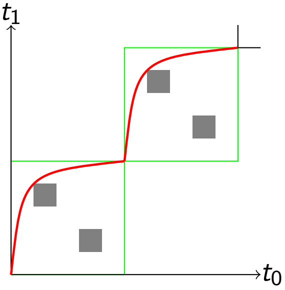
Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 → 0.1111) → 1.0111 → 0.1101 → 1.1101

↓

0.0101 ← 1.1111 ← 0.0111 ← 1.1101 ← 0.1111

Clean-memory model



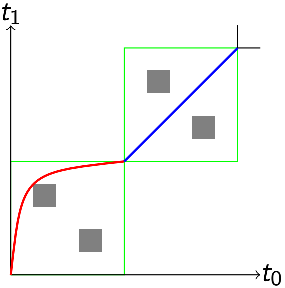
Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 —11→ 0.1111) —1→ 1.0111 —1→ 0.1101 ⟶ 1.1101

↓

0.0101 ⟵ 1.1111 ⟵ 0.0111 ⟵ 1.1101 ⟵ 0.1111

Clean-memory model

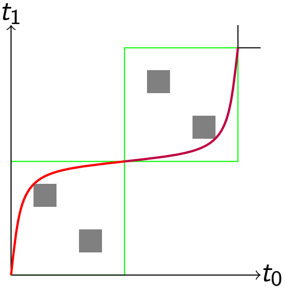


Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 —11→ 0.1111) —13→ 1.0111 —1→ 0.1101 ——→ 1.1101

↓

0.0101 ←—— 1.1111 ←—— 0.0111 ←—— 1.1101 ←—— 0.1111

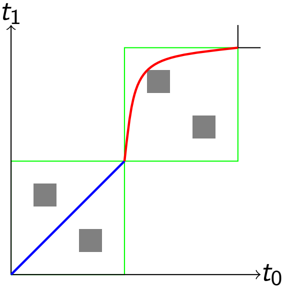


Iterated subdivision (fractal) of the protocol complex (round 2):

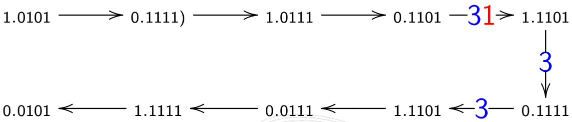
$$1.0101 \xrightarrow{11} 0.1111) \xrightarrow{13} 1.0111 \xrightarrow{12} 0.1101 \longrightarrow 1.1101$$

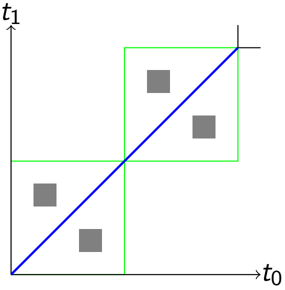
$$\downarrow$$

$$0.0101 \longleftarrow 1.1111 \longleftarrow 0.0111 \longleftarrow 1.1101 \longleftarrow 0.1111$$



Iterated subdivision (fractal) of the protocol complex (round 2):



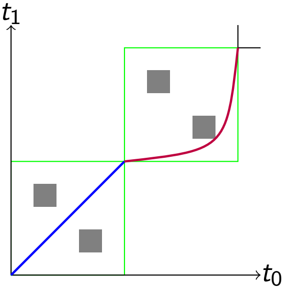


Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 → 0.1111) → 1.0111 → 0.1101 —31→ 1.1101

↓ 33

0.0101 ← 1.1111 ← 0.0111 ← 1.1101 ←3— 0.1111

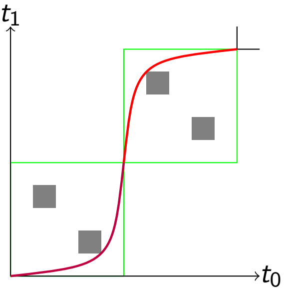


Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 ⟶ 0.1111) ⟶ 1.0111 ⟶ 0.1101 —31→ 1.1101

1.1101 —33↓ 0.1111

0.0101 ⟵ 1.1111 ⟵ 0.0111 ⟵ 1.1101 ←32— 0.1111

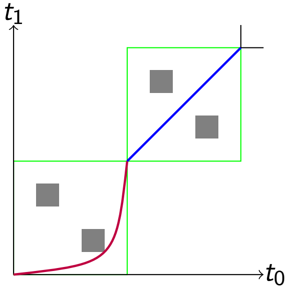


Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 ⟶ 0.1111) ⟶ 1.0111 ⟶ 0.1101 ⟶ 1.1101

↓

0.0101 ⟵2— 1.1111 ⟵2— 0.0111 ⟵21— 1.1101 ⟵ 0.1111

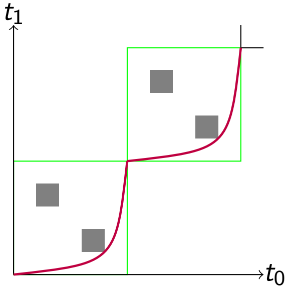


Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 → 0.1111) → 1.0111 → 0.1101 → 1.1101

↓

0.0101 ←2— 1.1111 ←23— 0.0111 ←21— 1.1101 ← 0.1111



Iterated subdivision (fractal) of the protocol complex (round 2):

1.0101 ⟶ 0.1111) ⟶ 1.0111 ⟶ 0.1101 ⟶ 1.1101

↓

0.0101 ⟵22— 1.1111 ⟵23— 0.0111 ⟵21— 1.1101 ⟵ 0.1111

Hence

Theorem

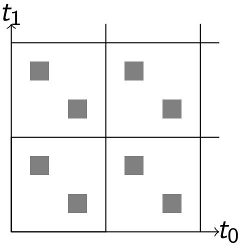
The clean memory model for n processes at round r produces a subdivided n simplex (up to some "flares" which do not affect $(n-1)$-connectedness)

(The flares are ruled out, classically, by the layered execution requirement)

- Clear relation with underlying geometric semantics
- All is fine, but is there a new result here? Not yet...

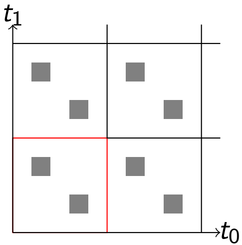
Example: same-memory model

Much more complicated! But fits in our framework perfectly



Example: same-memory model

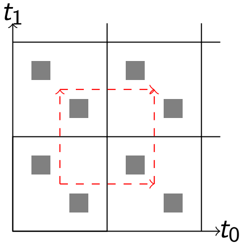
Much more complicated! But fits in our framework perfectly



$\rightarrow$ each block (1 unfolding) creates an $(n-1)$-connected complex

Example: same-memory model

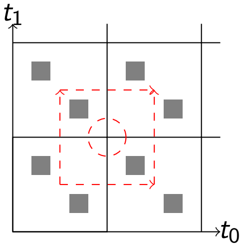
Much more complicated! But fits in our framework perfectly



→ each block (1 unfolding) creates an $(n-1)$-connected complex
→ glued under some recurrence relation

Example: same-memory model

Much more complicated! But fits in our framework perfectly



→ each block (1 unfolding) creates an $(n-1)$-connected complex
→ glued under some recurrence relation
→ whose relations make it a contractible scheme for pasting blocks

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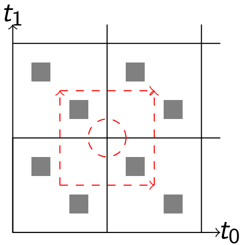
Much more complicated! But fits in our framework perfectly



→ each block (1 unfolding) creates an $(n-1)$-connected complex
→ glued under some recurrence relation
→ whose relations make it a contractible scheme for pasting blocks
→ hence (nerve lemma), creates an $(n-1)$-connected protocol complex! (not previously described, as this does not create an iterated subdivided simplex)

In general...: interval posets and schedules

Interval posets

- Let S be a set of closed intervals in $\mathbb{R}$ (i.e. of elements of the form $[a, b]$, a, b in $\mathbb{R}$)
- We define the partial order:

$$[a, b] \leqslant [c, d] \Leftrightarrow b \leqslant c$$

- $(S, \leqslant)$ is called an interval poset

- Are very well described, combinatorially
- For instance Fishburn's theorem (equivalence with (2+2)-free posets)
- And number of such posets on n elements is well known, example: 1,3,19,207,3451,... (this is A079144 on OEIS)

The dihomotopy classes of maximal paths, for the 1-round scan/update model for n processes, is in bijection with the interval posets on n elements.

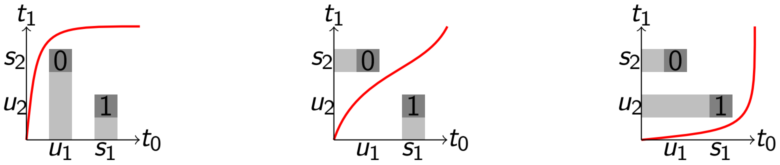
The bijection associates to each dihomotopy class $[p]$ the set of intervals in $[0, 1]$

$$(p \circ \pi_i)^{-1}([u_i, s_i])$$

$(i = 1, \ldots, n)$

Proof relies on the characterization of dihomotopy classes through alive matrices, hence dead matrices - recall condition on being dead, as some interval inequalities!

Example, in dimension 2



$[u_2, s_2] < [u_1, s_1]$ $[u_1, s_1], [u_2, s_2]$ $[u_2, s_2] < [u_1, s_1]$

What is the structure of the protocol complex now?

Extension order on posets

Let $(S_1 \leqslant_1)$ and $(S_2, \leqslant_2)$ be two partial order on some sets $S_1 \subseteq S_2$. We say that $\leqslant_1 \Rightarrow \leqslant_2$ if $\forall s, t \in S_1$, $s \leqslant_1 t \Rightarrow s \leqslant_2 t$.

When $S_1 = S_2$, this is the linearization order.

Importance of the extension order for our purpose

Let $\leqslant_1$ and $\leqslant_2$ be interval orders on the same set of cardinal $n+1$. If $\leqslant_1$ is a linearization of $\leqslant_2$ then the corresponding n-simplexes share a common $(n-1)$ face.

In fact, the face poset of the protocol complex is given by the extension order on interval posets up to n elements

Structure of the protocol complex

Corollary

The protocol complex for scan/update in dimension n, for one round, is homotopy equivalent to the order complex for the extension order on interval posets up to n elements.

(since the order complex of the face poset is just the barycentric subdivision)

Theorem

The protocol complex for the scan/update model, in dimension n, for one round, is an $(n-1)$-connected simplicial set. It is a subdivision of $\Delta[n]$ plus some extra contractible "flares".

The flares are ruled out, classically, by the layered execution requirement

19 maximal alive matrices (for 53 dead ones)

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	0	1	0	0	1	0	0	1	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0	0	1	0
1	0	0	1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	1	0	0	1	0
0	1	0	0	1	0	0	1	0	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	1	0	0	0	0	1
0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	0	0	1			
1	0	0	0	0	1	0	0	1	0	0	1			
0	0	1	0	0	1	0	0	1	0	0	1			
1	0	0	1	0	0	1	0	0	0	1	0			
0	0	1	0	0	1	0	0	1	0	0	1			
0	1	0	1	0	0	0	1	0	0	1	0			

Reorganizing things a bit...

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	0	1	0	0	1	0	0	1	0
4	2	0	3	3	0	2	3	1	2	4	0	1	4	1

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0	0	1	0
1	0	0	1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	1	0	0	1	0
4	1	1	3	2	1	3	1	2	2	2	2	1	3	2

0	1	0	0	1	0	0	1	0	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	1	0	0	0	0	1
0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	4	2	1	2	3	0	3	3	4	0	2	3	0	3

0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	0	0	1	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	0	1	0
2	1	3	2	0	4	1	1	4	0	2	4

1 of symmetry type (2,2,2)

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	0	1	0	0	1	0	0	1	0
4	2	0	3	3	0	2	3	1	2	4	0	1	4	1

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0	0	1	0
1	0	0	1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	1	0	0	1	0
4	1	1	3	2	1	3	1	2	2	2	2	1	3	2

0	1	0	0	1	0	0	1	0	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	1	0	0	0	0	1
0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	4	2	1	2	3	0	3	3	4	0	2	3	0	3

0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	0	0	1	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	0	1	0
2	1	3	2	0	4	1	1	4	0	2	4

6 of symmetry type (3,2,1)

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	0	1	0	0	1	0	0	1	0
4	2	0	3	3	0	2	3	1	2	4	0	1	4	1

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
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0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0	0	1	0
1	0	0	1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	1	0	0	1	0
4	1	1	3	2	1	3	1	2	2	2	2	1	3	2

0	1	0	0	1	0	0	1	0	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
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0	0	1	0	0	1	0	0	1	1	0	0	0	0	1
0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	4	2	1	2	3	0	3	3	4	0	2	3	0	3

0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	0	0	1	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	0	0	1
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0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	0	1	0
2	1	3	2	0	4	1	1	4	0	2	4

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	0	1	0	0	1	0	0	1	0
4	2	0	3	3	0	2	3	1	2	4	0	1	4	1

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
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1	0	0	1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	1	0	0	1	0
4	1	1	3	2	1	3	1	2	2	2	2	1	3	2

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0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
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0	0	1	0	0	1	0	0	1	1	0	0	0	0	1
0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	4	2	1	2	3	0	3	3	4	0	2	3	0	3

0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	0	0	1	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	0	1	0
2	1	3	2	0	4	1	1	4	0	2	4

3 of symmetry type (4,1,1)

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	0	1	0	0	1	0	0	1	0
4	2	0	3	3	0	2	3	1	2	4	0	1	4	1
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
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1	0	0	1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	1	0	0	1	0
4	1	1	3	2	1	3	1	2	2	2	2	1	3	2
0	1	0	0	1	0	0	1	0	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	1	0	0	1
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0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	4	2	1	2	3	0	3	3	4	0	2	3	0	3
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1	0	0	0	0	1	0	0	1	0	0	1			
0	0	1	0	0	1	0	0	1	0	0	1			
1	0	0	1	0	0	1	0	0	0	1	0			
0	0	1	0	0	1	0	0	1	0	0	1			
0	1	0	1	0	0	0	1	0	0	1	0			
2	1	3	2	0	4	1	1	4	0	2	4			

6 of symmetry type (4,2,0)

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	0	1	0	0	1	0
1	0	0	1	0	0	0	0	1	1	0	0	0	0	1
1	0	0	0	1	0	0	1	0	0	1	0	0	1	0
4	2	0	3	3	0	2	3	1	2	4	0	1	4	1

0	1	0	0	1	0	0	1	0	0	1	0	0	1	0
1	0	0	1	0	0	1	0	0	1	0	0	1	0	0
0	0	1	0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	1	0	0	0	1	0
1	0	0	1	0	0	0	0	1	0	0	1	0	0	1
1	0	0	0	1	0	1	0	0	0	1	0	0	1	0
4	1	1	3	2	1	3	1	2	2	2	2	1	3	2

0	1	0	0	1	0	0	1	0	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	1	0	0	1	0	0
0	1	0	0	0	1	0	0	1	0	0	1	0	0	1
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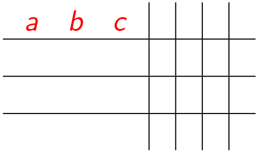
0	0	1	0	0	1	0	0	1	1	0	0	0	0	1
0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	4	2	1	2	3	0	3	3	4	0	2	3	0	3

0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	0	0	1	0	0	1	0	0	1
0	0	1	0	0	1	0	0	1	0	0	1
1	0	0	1	0	0	1	0	0	0	1	0
0	0	1	0	0	1	0	0	1	0	0	1
0	1	0	1	0	0	0	1	0	0	1	0
2	1	3	2	0	4	1	1	4	0	2	4

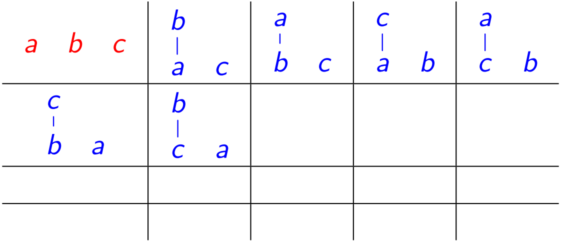
Corresponding to the labelled interval posets on 3 elements (19 of them)

(2,2,2)



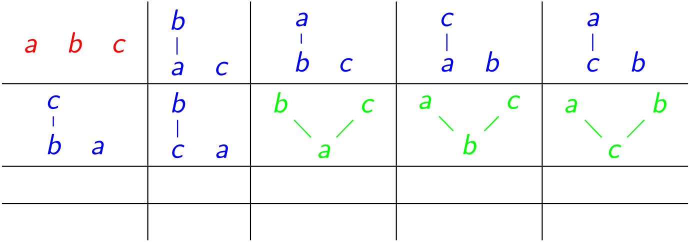
Corresponding to the labelled interval posets on 3 elements (19 of them)

(3,2,1)



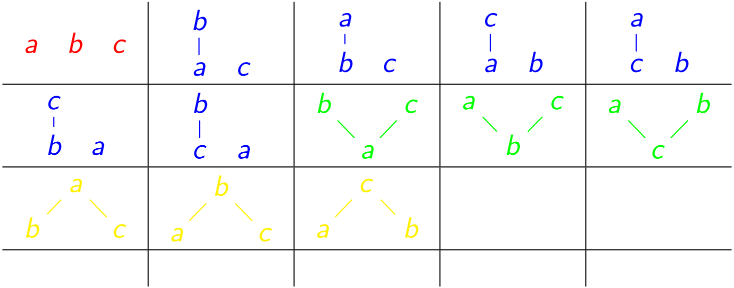
Corresponding to the labelled interval posets on 3 elements (19 of them)

(3,3,0)



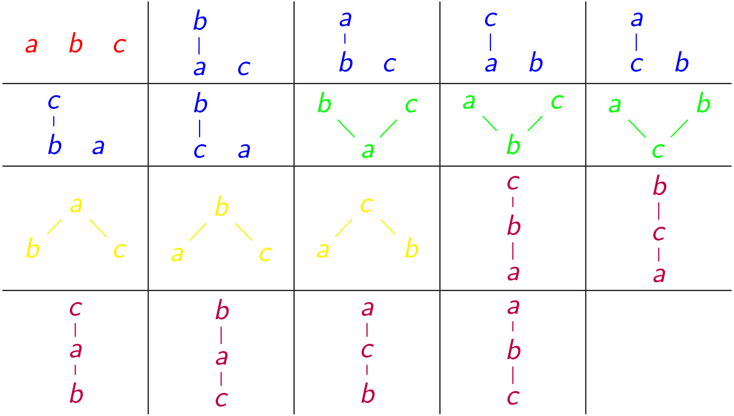
Corresponding to the labelled interval posets on 3 elements (19 of them)

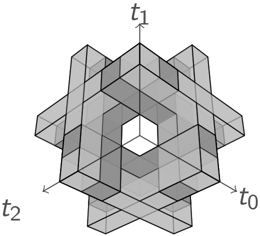
(4,1,1)



Corresponding to the labelled interval posets on 3 elements (19 of them)

(4,2,0)



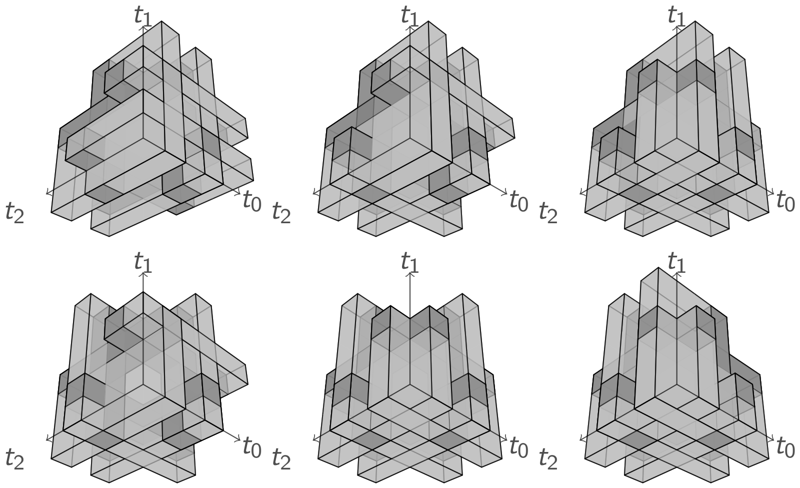


Hasse diagram of the corresponding interval poset:

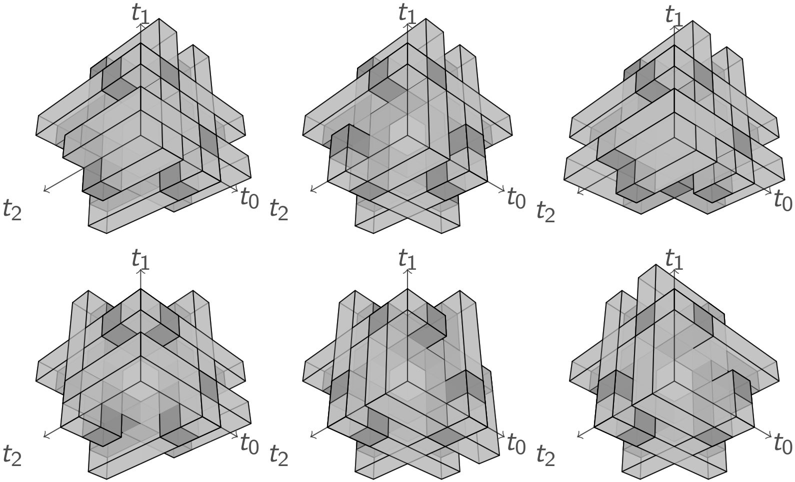
$$[u_0, s_0] \quad [u_1, s_1] \quad [u_2, s_2]$$

(let us call $a = [u_0, s_0]$, $b = [u_1, s_1]$, $c = [u_2, s_2]$ for the sequel)

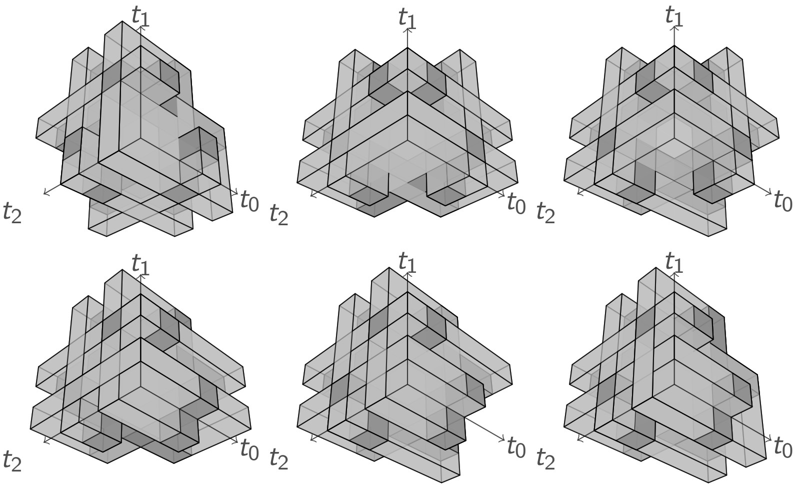
Example: in dimension 3 (the 18 other schedules)



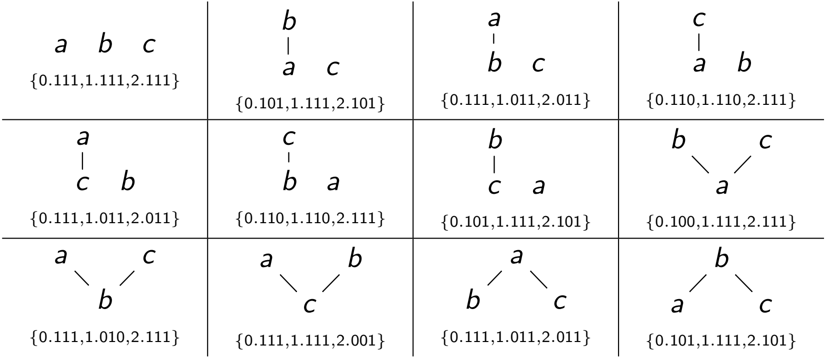
Example: in dimension 3 (the 18 other schedules)



Example: in dimension 3 (the 18 other schedules)

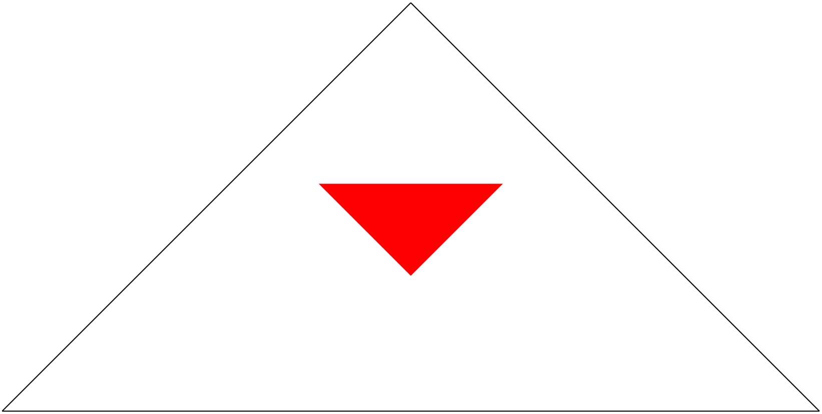


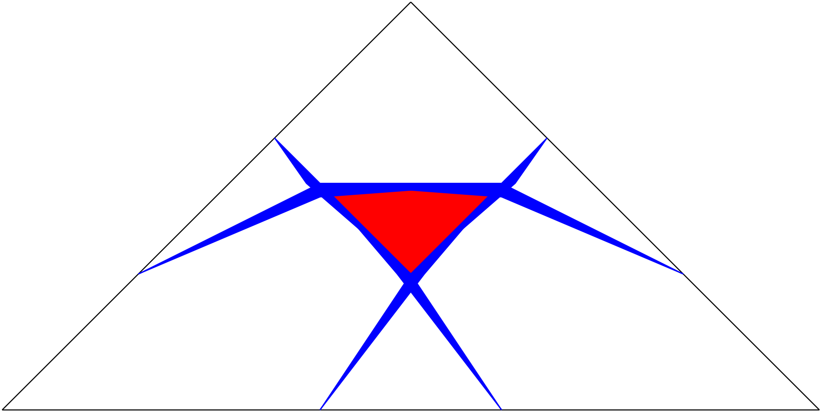
Each interval can be interpreted in terms of "knowledge", hence the structure of the protocol complex...



etc.

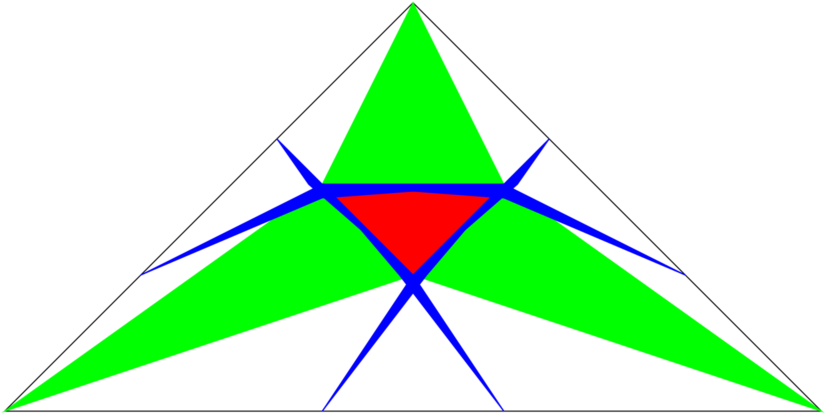
Construction of the protocol complex



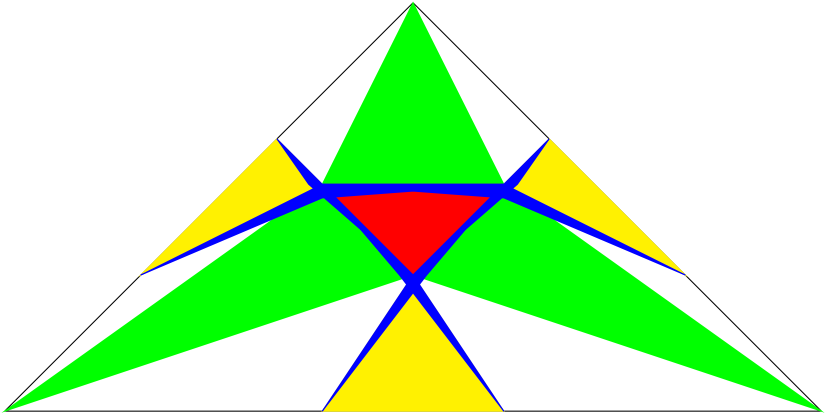


These are ruled out under the layered execution model

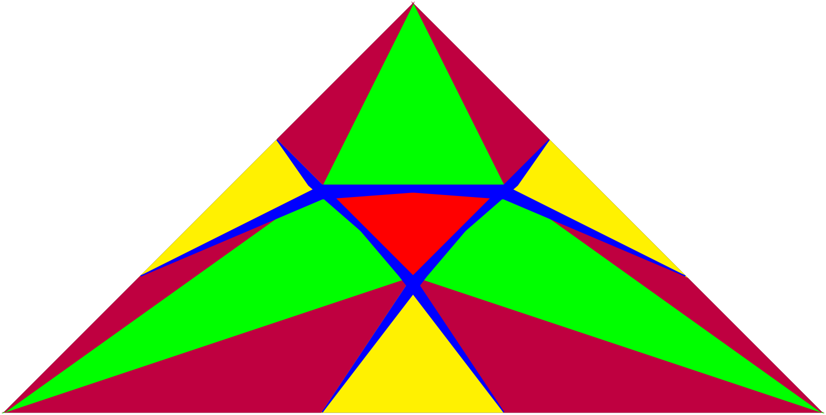
Construction of the protocol complex



Construction of the protocol complex

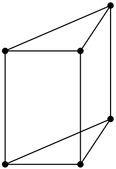


Construction of the protocol complex



Trace spaces: prodsimplicial structure

- A prod-simplicial space is just a space made up of simplices, and products of simplices, glued together along their faces (natural generalization of cubical and simplicial sets)



Trace spaces: prodsimplicial structure

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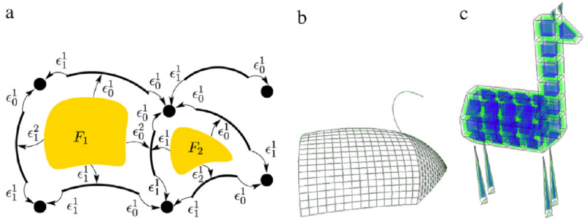
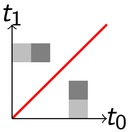
- Example:



The prodsimplicial structure of trace spaces

Each matrix of $\mathcal{C}$ represents a prod-simplex, product of one n-simplex per line, n=number of 1 per line minus 1...

Recall:

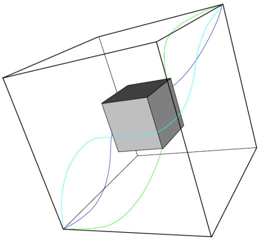


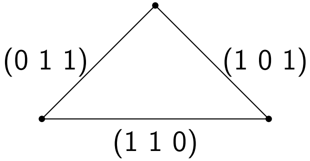
$$M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

product of 2 0-simplices = point!

The prodsimplicial structure of trace spaces

Each matrix of $\mathcal{C}$ represents a prod-simplex, product of one n-simplex per line, n=number of 1 per line minus 1...



- $\mathcal{D}(X)(0,1) = \{(111)\}$
- $\mathcal{C}(X)(0,1) = \{(110),(101),(011)\}$
- 

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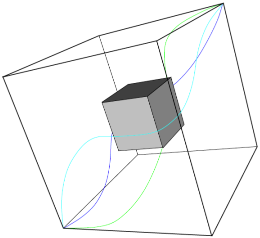
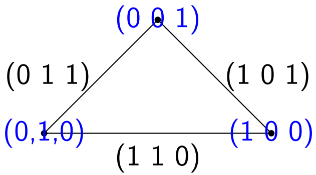


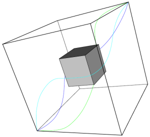
- $\mathcal{C}(X)(0,1) = \{(110), (101), (011)\}$
- and common faces are meet of matrices

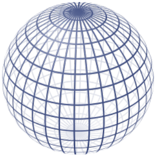


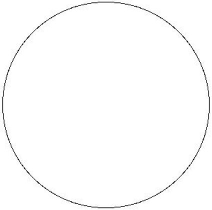
The prodsimplicial structure of trace spaces

Each matrix of $\mathcal{C}$ represents a prod-simplex, product of one n-simplex per line, n=number of 1 per line minus 1...

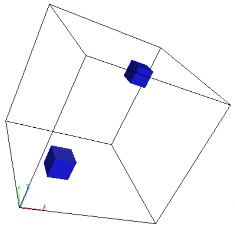
- $\mathcal{C}(X)(0,1) = \{(110), (101), (011)\}$
- connected, not simply-connected (reflecting the fact that $\pi_2(X) = \mathbb{Z}$)



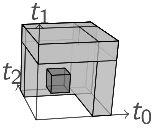




A more intricate example

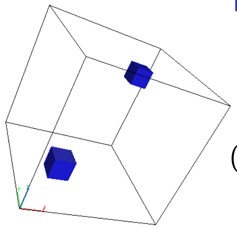


- $\mathcal{D}(X)(0,1) = \left\{ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \right\}$

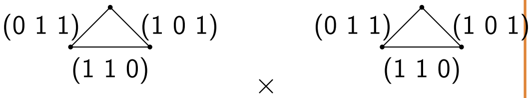


- $\mathcal{C}(X)(0,1) = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ldots \right\}$

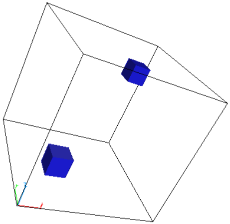
A more intricate example

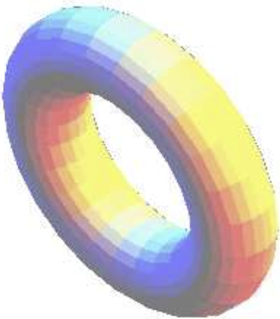


- $\mathcal{C}(X)(0,1) = \left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ldots \right\}$



- $\mathcal{C}(X)(0,1) =$
$$\left\{ \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \ldots \right\}$$





(π_1 is $\mathbb{Z} \times \mathbb{Z}$)

In short: theorems

Theorem

The prodsimplicial set corresponding to the scan/update model, in any dimension n, for one round, is discrete. Its cardinal is the number of interval posets on n elements.

Compare with:

Theorem

The protocol complex for the scan/update model, in dimension n, for one round, is an $(n-1)$-connected simplicial set. It is a subdivision of $\Delta[n]$ plus some extra contractible "flares".

In short: conjecture

Conjectural construction of protocol complexes

The protocol complex is homotopy equivalent to the transversal hypergraph made of dead matrices (a hypergraph is in particular a simplicial set).

For $n = 2$ we saw that; for $n = 3$, the transversal hypergraph is a 11 dimensional simplicial set; for any n it is of dimension $n(n-1)^2 - 1$.

Sort of duality between prodsimplicial representation and the protocol complex one?

Conclusion and future work

- Lots of experiments and lots of mathematics to be investigated yet on trace spaces...
- Applications to more subtle (and less combinatorial) models for protocols, in particular the "same memory model", and more intricate synchronisation primitives (test&set, fetch&add etc.)
- Extension to randomized algorithms: random simplicial sets! (account for possibility of consensus)
- Logical interpretation of these 2 frameworks, simplicial, and directed
- etc.

Thanks for your attention!

`http://acat.lix.polytechnique.fr`