The Transient Behavior of Long Walks and Applications

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based on joint work with B. Charron-Bost and M. Függer

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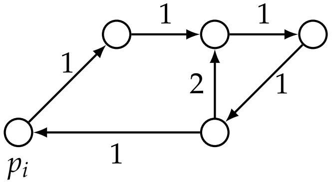
Overview

1. Problem Statement
   - Example: Network Synchronizer
   - Transient Behavior
   - Example: Link Reversal
   - Non-Weighted Digraphs
2. Transience Bounds
   - Previous Transience Bounds
   - Repetitive and Explorative Bounds

Synchronizer Definition

- Consider a message-passing network of N fault-free processes
- Described by a strongly connected digraph
- The message delay on every link is constant
- Processes run a wait-for-all synchronizer
- Process p_i sends its initial message at time T_i
- What's the time behavior of this system?

Initial Message at Time 0



- First assume that all initial messages are sent at time $T_i = 0$
- Pick some process p_i
- Times at which p_i sends messages: $0, 1, 2, 3, 5, 6, 7, 9, 10, 11, 13 \ldots$

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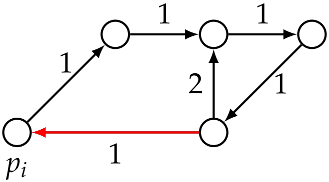

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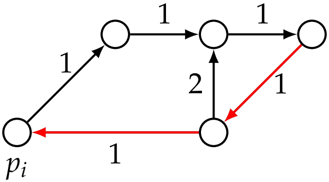

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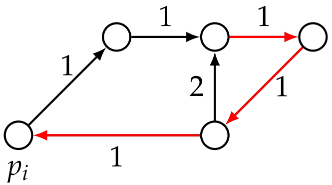

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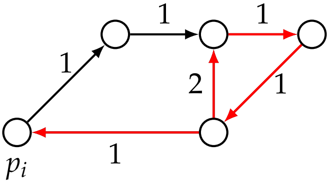

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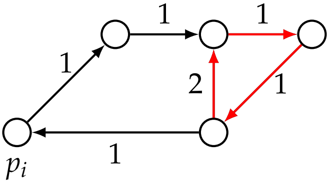

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Recursion Formula

- Recursion:

$$t_i(n+1) = \max_{j \to i} \; t_j(n) + d(j,i)$$

  with $t_i(0) = T_i$ and $d(j,i) =$ message delay from p_j to p_i
- $t_i(n) =$ greatest weight of walks of length n ending in i
- “max-plus” recursion

Max-Plus Linearity

- Sequence of vectors $x(n)$ defined by a recursion of the form

$$x_i(n+1) = \max_j \left(x_j(n) + A_{i,j}\right)$$

  where $A_{i,j} = -\infty$ is possible
- Solution of recursion is $x(n) = A^{\otimes n} \otimes x(0)$
- These systems are linear if we consider the matrix multiplication

$$(A \otimes B)_{i,j} = \max_k \left(A_{i,k} + B_{k,j}\right)$$

- Fact: $\left(A^{\otimes n}\right)_{i,j}$ = largest length n weight from i to j

ÉCOLE POLYTECHNIQUE

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   - Previous Transience Bounds
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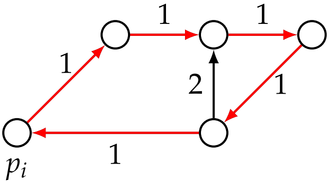
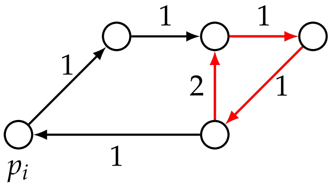
Critical Cycles Dominate



- One cycle with mean weight $= 1$
- Another with mean weight $= 4/3$
- The higher mean weight dominates
- Limit-average of time between messages at all processes: $4/3$

Critical Cycles Dominate



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Maximum Weights Between Two Nodes

- Periodic with "linear defect":
  $a(n+p) = a(n) + p \cdot \lambda$
- Fact: All these sequences become periodic if the graph is strongly connected. (Cohen et al. '83)

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Full Reversal Algorithm [Gafni & Bertsekas, 1981]

- Input: oriented connected graph G_0 and a subset D of nodes
- FR rule: a sink not in D reverses all its (incoming) links
- Execution: discrete time base $\mathcal{T} = \mathbb{N}$
- Greedy execution: at every time step, all nodes able to apply the FR rule do so
- Work vector:
  $w_i(t) =$ #times that node i applies the FR rule up to time t

Full Reversal Algorithm [Gafni & Bertsekas, 1981]

Theorem (Gafni & Bertsekas, 1981)

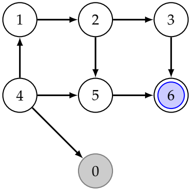
In every greedy execution, the work vector w is eventually periodic, i.e., there are $p \in \mathcal{T}$ and $\omega \in \mathbb{N}$ such that

$$\exists t_0, \forall i \in V(G), \forall t \geq t_0, \;\; w_i(t+p) = w_i(t) + \omega$$

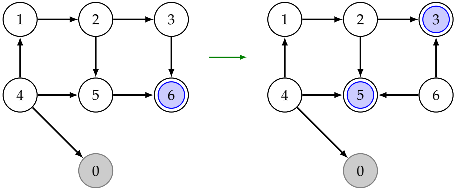
Furthermore, if $D \neq \emptyset$, then every execution terminates, i.e., $p = 1, \omega = 0$.

Applications: routing, leader election, resource allocation, …

Full Reversal



Full Reversal



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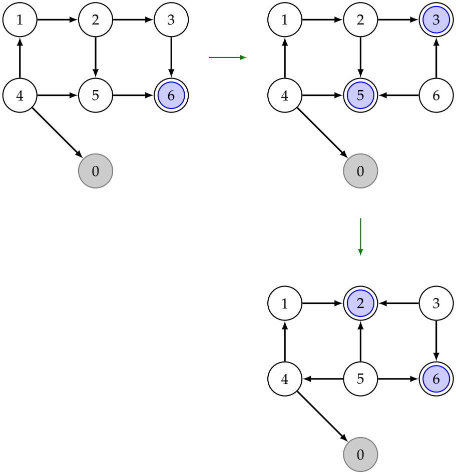

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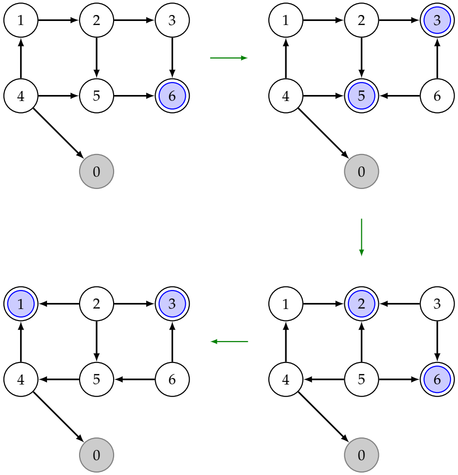

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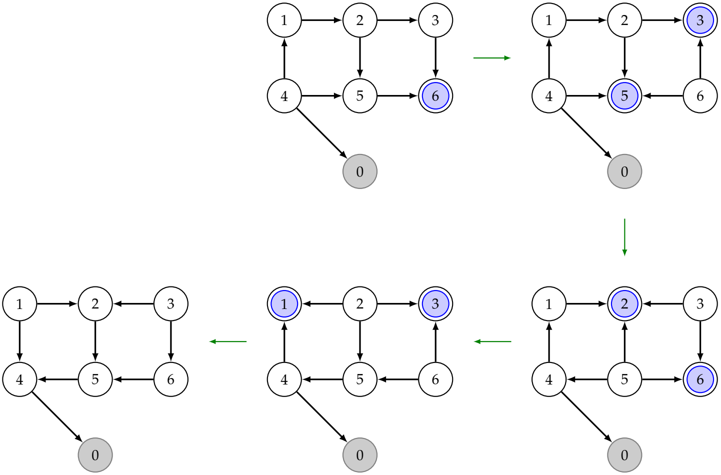


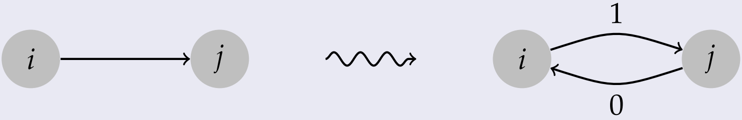
Full Reversal is Min-Plus Linear

Theorem (Charron-Bost, Függer, Welch, Widder 2011)

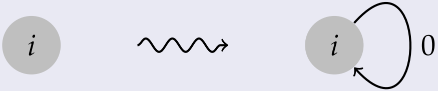
The work vector w of a greedy FR execution fulfills a min-plus recursion.

Full Reversal is Min-Plus Linear

Proof.



$i \in D$:



$\square$

Applications of Max-Plus

Other systems with a max-plus recursion include:

- Transportation networks (train schedules, ...)
- Manufacturing plants
- Cyclic scheduling
- Timed event graphs

Our bounds give design guidelines for small transient phases, because they include graph parameters.
E.g., $O(N)$ if the support is a tree.

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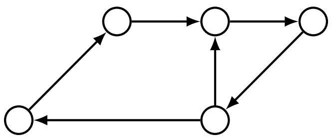
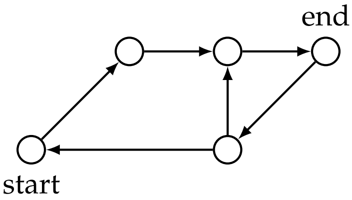
The Lengths Between Two Nodes



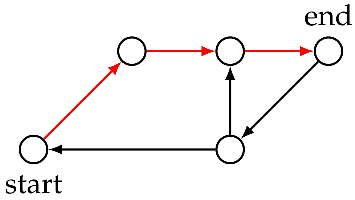
- Pick two nodes in a directed graph
- Form the following sequence: for every n, write “1” if there is a walk between the nodes that has length n, and write “0” otherwise.

The Lengths Between Two Nodes



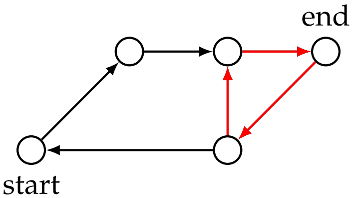
- Let's start at $n = 0$
- $0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, \ldots$

The Lengths Between Two Nodes



- Let's start at $n = 0$
- $0, 0, 0, {\color{red}1}, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, \ldots$

The Lengths Between Two Nodes



- Let's start at $n = 0$
- $0, 0, 0, \textcolor{red}{1}, 0, 0, \textcolor{red}{1}, 0, 1, \textcolor{red}{1}, 0, 1, \textcolor{red}{1}, 1, \ldots$

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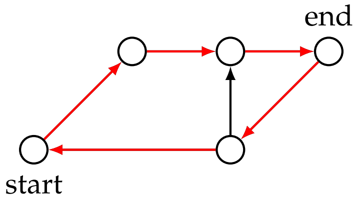
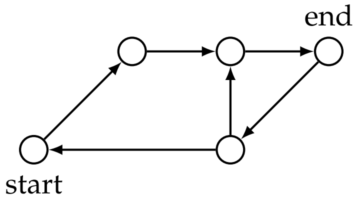


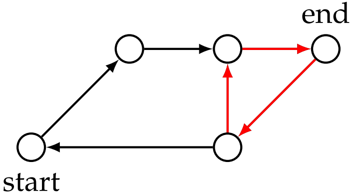
- Let's start at $n = 0$
- $0, 0, 0, \textcolor{red}{1}, 0, 0, 1, 0, \textcolor{red}{1}, 1, 0, 1, 1, \textcolor{red}{1}, \ldots$

The Lengths Between Two Nodes



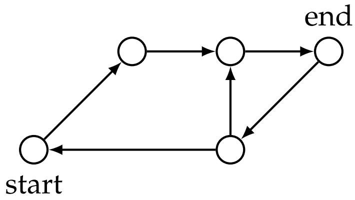
- Let's start at $n = 0$
- $0, 0, 0, 1, 0, 0, 1, 0, 1, 1, 0, 1, 1, 1, \ldots$
- Fact: This sequence becomes periodic.
- Main Question: How long is the transient phase?

The Lengths Between Two Nodes



- First Question: What is the period?
- Every cycle you meet along the way adds a "$+L$" pattern, where L is its length.
- Example: $L = 3$
  $0, 0, 0, {\color{red}1}, 0, 0, {\color{red}1}, 0, 1, {\color{red}1}, 0, 1, {\color{red}1}, 1, \ldots$

The Lengths Between Two Nodes



- If strongly connected:
  $X + \sum_C k_C \cdot L_C$
- Candidate for period (think $X = 0$):
  GCD of cycle lengths (Bézout)
- Indeed, period = GCD ("cyclicity")
- Fact: The transient of an eventually periodic sequence is independent of the considered period.

Wielandt's Bound

- index of a graph = largest transient phase between two nodes
- N = number of nodes in the graph

Theorem (Wielandt; Math. Z. '50 / Schwarz; Cz. Math. J. '70)

The index of a strongly connected digraph is at most $(N-1)^2+1$.

Bounds Including Graph Parameters

- Dulmage and Medelsohn (Illinois J. Math. '64): included girth g = shortest cycle length
- Schwarz (Cz. Math. J. '70): included cyclicity γ = GCD of cycle lengths

Theorem (Kim; LAA '79)

The index of a strongly connected digraph is at most
$N + g \cdot \left(\left\lfloor \frac{N}{\gamma} \right\rfloor - 2 \right)$.

Nodes on Maximum Mean Cycles

Theorem (Merlet, N., Schneider, Sergeev, 2013)

Almost all bounds on the index of unweighted digraphs extend to weighted digraphs for the transients of nodes on maximum mean weight cycles.

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Previous Transience Bounds

- Even and Rajsbaum (STOC '90)
  - transience bound for an application (network synchronizer)
  - careful study of the proof gives a more general bound
  - in their special case: $O(N^3)$
- Hartmann and Arguelles (Math. Oper. Res. '99)
  - of the form $\max\left\{B'_c\,,\,2N^2\right\}$
  - inherently quadratic, i.e., $\Omega(N^2)$
- Bouillard and Gaujal (INRIA RR '00)
  - worst case: exponential in N
  - can also be linear
  - Akian et al. '05 gave a refinement

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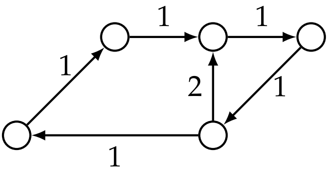
Critical Cycles

- A closed walk is critical if it has maximum weight-to-length ratio λ.
- Subgraph induced by critical closed walks: critical subgraph
- Fact: Every closed walk formed out of edges of critical walks (= in the critical subgraph) is also critical.



$\lambda = 4/3$

- By subtracting λ from all edge weights:
  WLOG $\lambda = 0$, i.e., the sequence is periodic without linear defect

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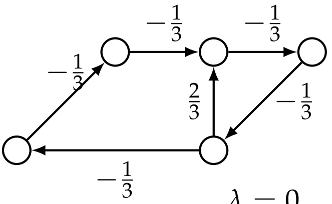

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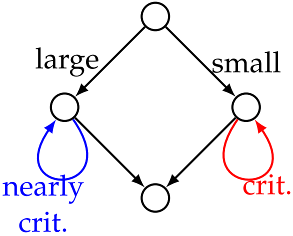
Critical Bound



- Δ = largest weight
  δ = smallest weight
- λ_{nc} = largest weight-to-length ratio outside of critical subgraph
- $\|x(0)\|$ = maximum difference between entries in $x(0)$

Lemma

For $n \geq B_c$, every length n maximum weight walk visits the critical subgraph.

Always: $B_c \leq \max\left\{N\,,\ \left(\|x(0)\| + (N-1)\cdot(\Delta-\delta)\right)/(\lambda-\lambda_{nc})\right\}$

Realizers

- π = LCM of cycle lengths

A walk $\tilde{W}$ between the two nodes is a *B*-realizer for node i if it has maximum weight among all walks W with the following properties:

- W starts from node i
- $\ell(W) \geq B$
- $\ell(W) \equiv \ell(\tilde{W}) \pmod{\pi}$

Lemma

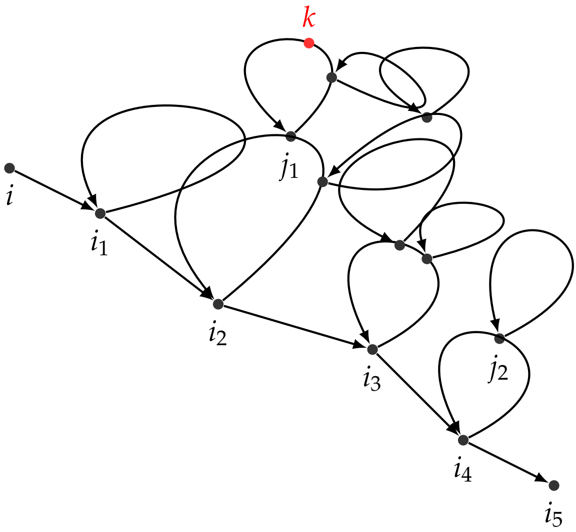
If, for every attainable $n \geq B$ and every i, there exists a B-realizer of length n, then B is an upper bound on the length of the transient phase.

Proof Strategy

Given: attainable n

1. Let W be a B-realizer of length congruent to n.
2. **Critical bound**: If $B \geq \boxed{B_c}$ then W visits a critical node k.
3. **Walk reduction**: Choose a divisor d of π. Remove critical subcycles from W until reduced walk $\hat{W}$ is short enough, but still $\ell(\hat{W}) \equiv \ell(W) \pmod{d}$ and contains node k.
4. **Pumping**: If $n \geq \boxed{B_d}$ we can add critical cycles at k to $\hat{W}$ until its length is equal to n. The new walk is still a realizer, because the weight did not change.
5. $B = \max\{B_c, B_d\}$

Walk Reduction



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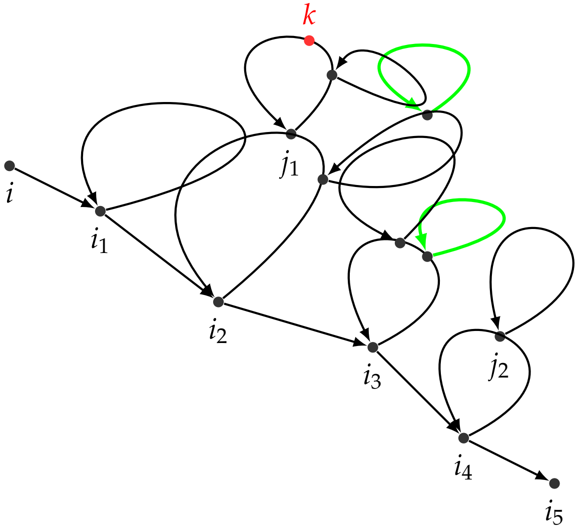

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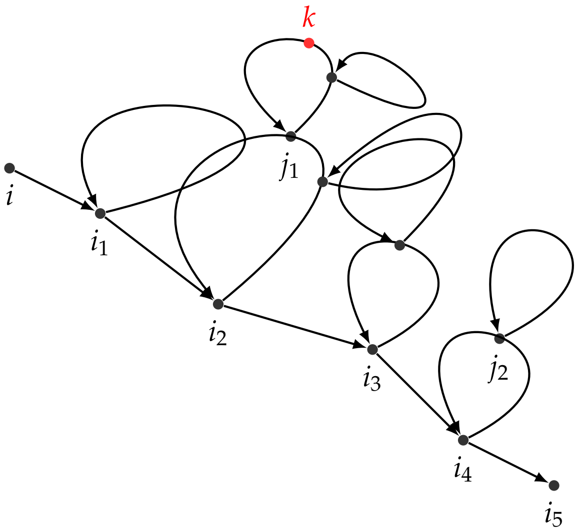

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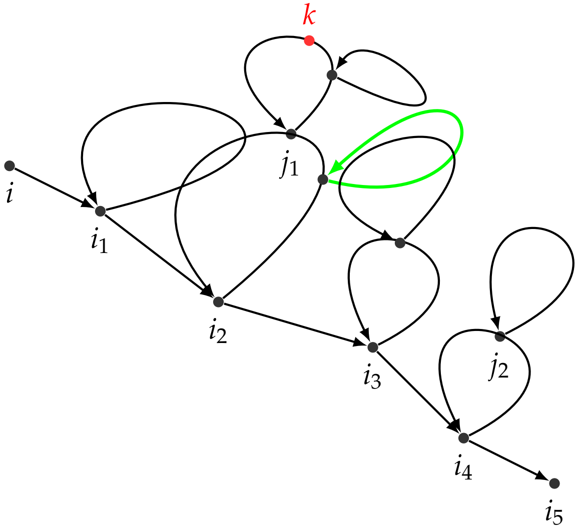

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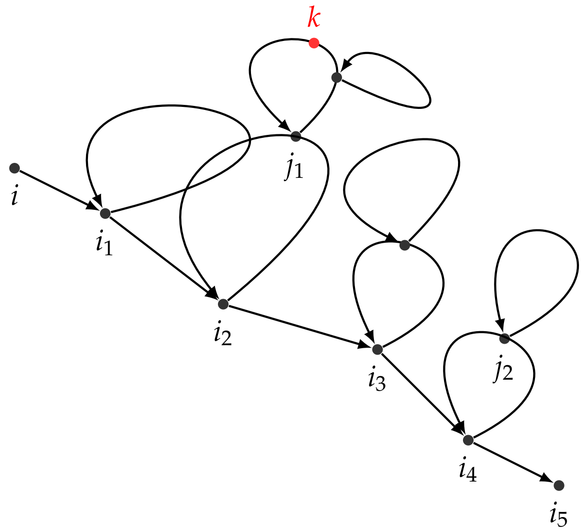

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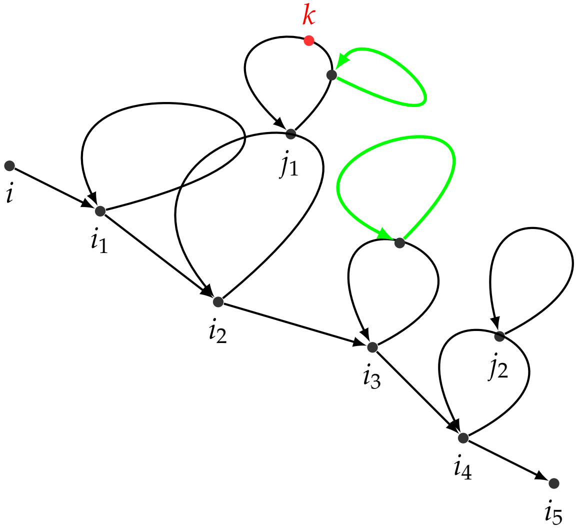

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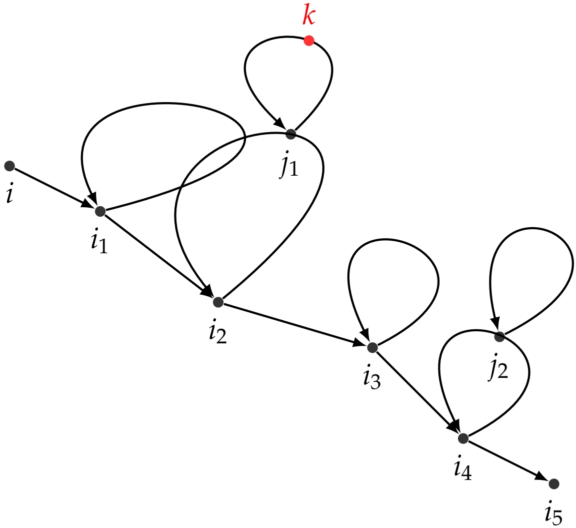

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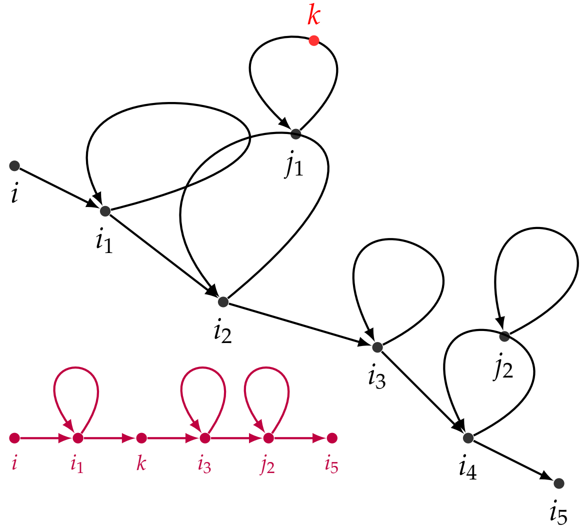


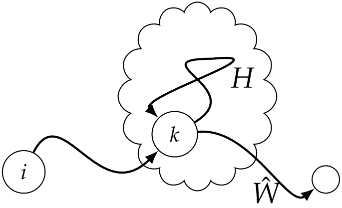
Length of Reduced Walk

Lemma

Every collection of d integers contains a non-empty subcollection whose sum is divisible by d.

$$\begin{aligned}\Rightarrow \ell(\hat{W}) &\leq (d-1)\cdot N + (d+1)\cdot(N-1)\\ &= (d-1) + 2d\cdot(N-1)\end{aligned}$$

Explorative Bound: $d = \gamma(H)$

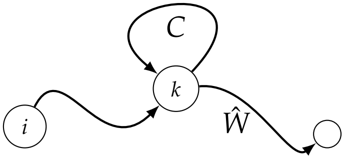


- $H = k$'s critical s.c.c.
- Choose $d = \gamma(H)$ = cyclicity of H: $l(\hat{W}) \leq (\gamma(H) - 1) + 2\gamma(H) \cdot (N - 1)$
- $\hat{\gamma}$ = largest cyclicity of critical s.c.c.'s
  $\widehat{ind}$ = largest index of critical s.c.c.'s

Theorem (Charron-Bost, Függer, N.)

The length of the transient phase is at most
$$\max\left\{B_c\,,\ (\hat{\gamma} - 1) + 2\hat{\gamma} \cdot (N - 1) + \widehat{ind}\right\}.$$

Repetitive Bound: $d = g(H)$



- $H = k$'s critical s.c.c.
- By repeating a connecting closed walk in H: WLOG k lies on a critical cycle of length $g(H)$.
- Choose $d = g(H) =$ girth of H: $l(\hat{W}) \leq (g(H) - 1) + 2g(H) \cdot (N - 1)$
- $\hat{g} =$ largest girth of critical s.c.c.'s

Theorem (Charron-Bost, Függer, N.)

The length of the transient phase is at most
$\max\left\{B_c\,,\ (\hat{g} - 1) + 2\hat{g} \cdot (N - 1)\right\}$.

Extensions

- In the critical bound: separation line between the interior and exterior of the critical subgraph
- Compare critical subgraph to exterior
- Merlet, N., Sergeev: Make exterior graph smaller, bigger “region of influence” of critical subgraph
- Better “critical bound” with the rest of the bound untouched
- Includes more graph parameters

THANKS!