The Topology of Asynchronous Byzantine Colorless Tasks

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Overview

Basic concepts of combinatorial topology

Duality between combinatorial & continuous mechanisms

First *characterization* of tasks that can be solved in the asynchronous Byzantine model

Road Map

Colorless Tasks

Operational Model

Combinatorial Model

Building Blocks

Crash Failure Solvability

Byzantine Failure Solvability

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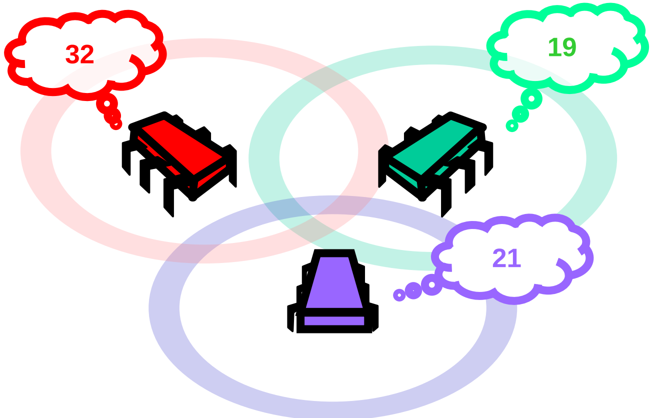
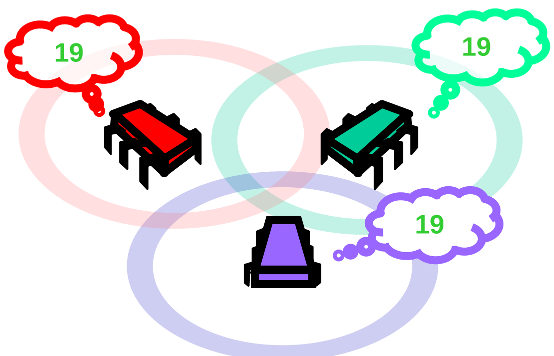
Tasks

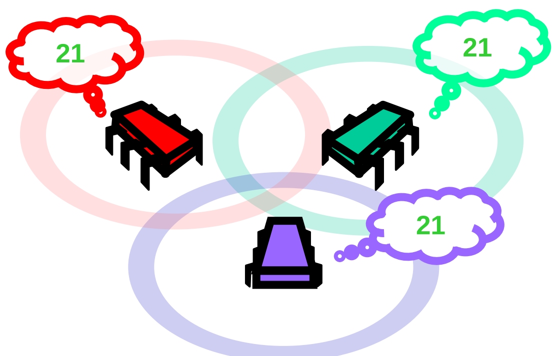


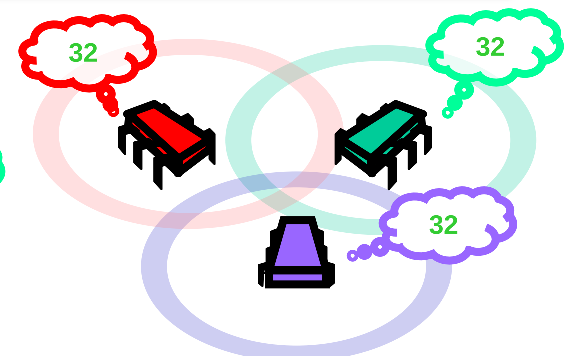
Possible set of *input* values

Finite computation

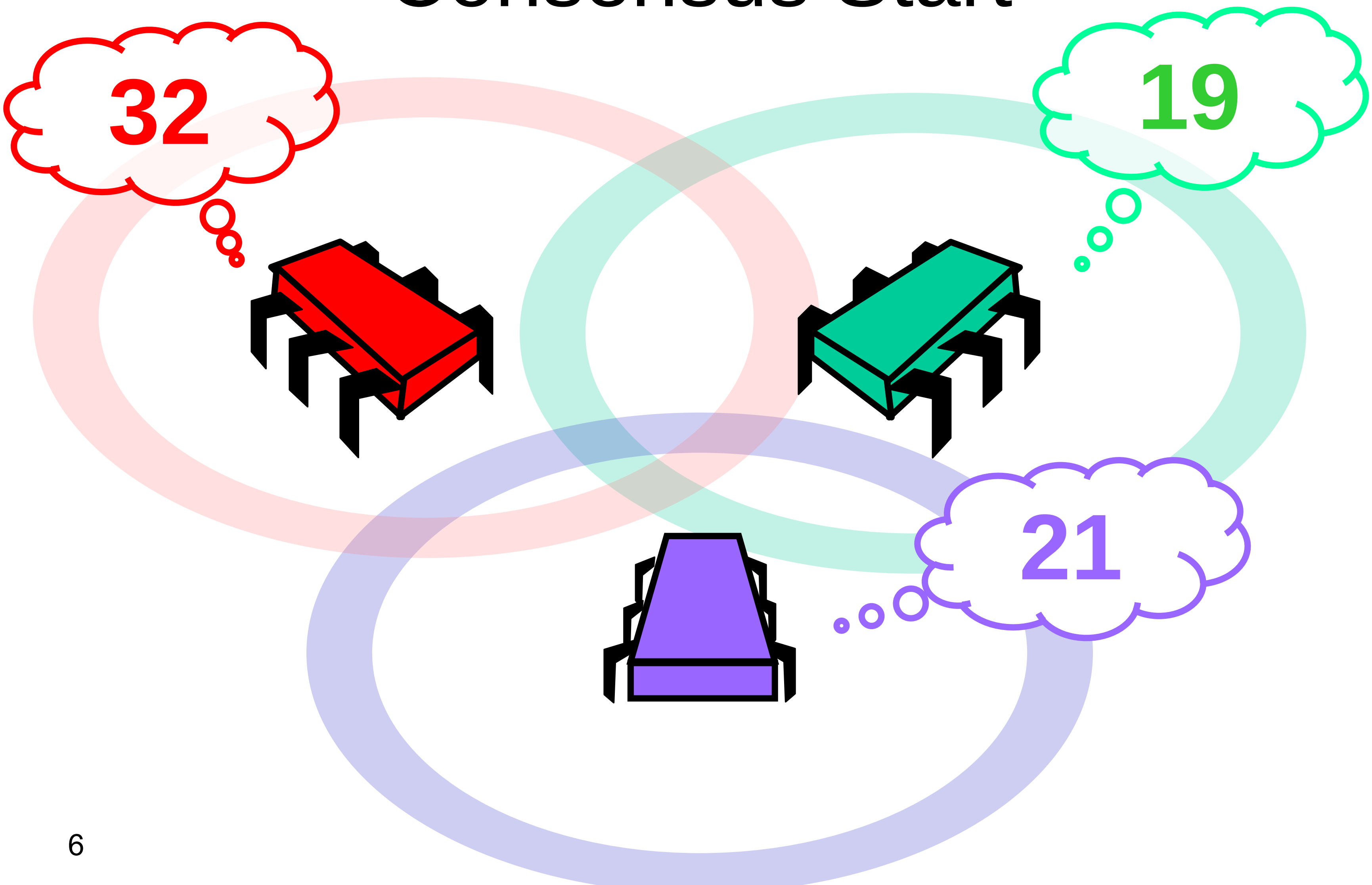
Possible set of *output* values



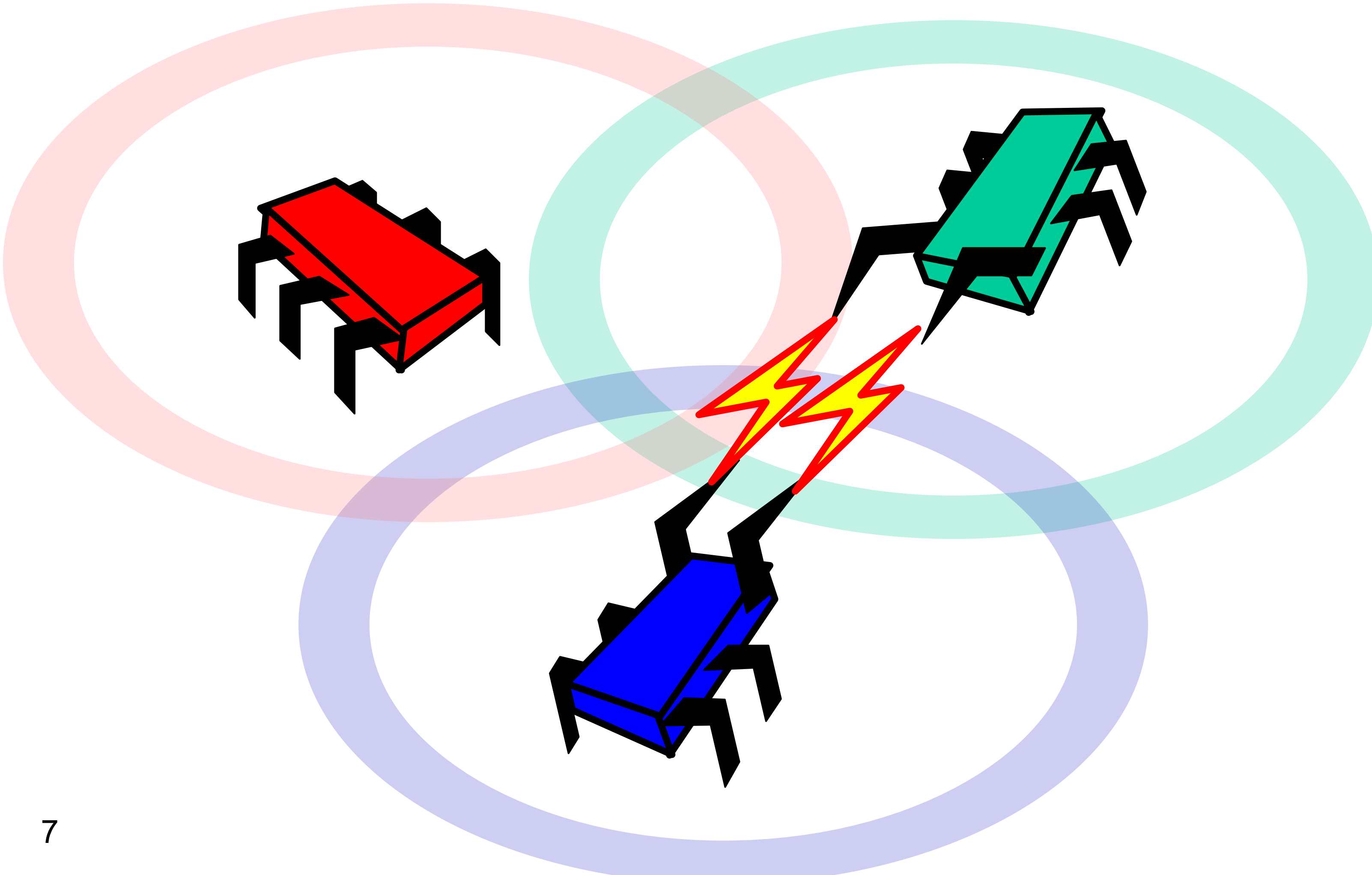




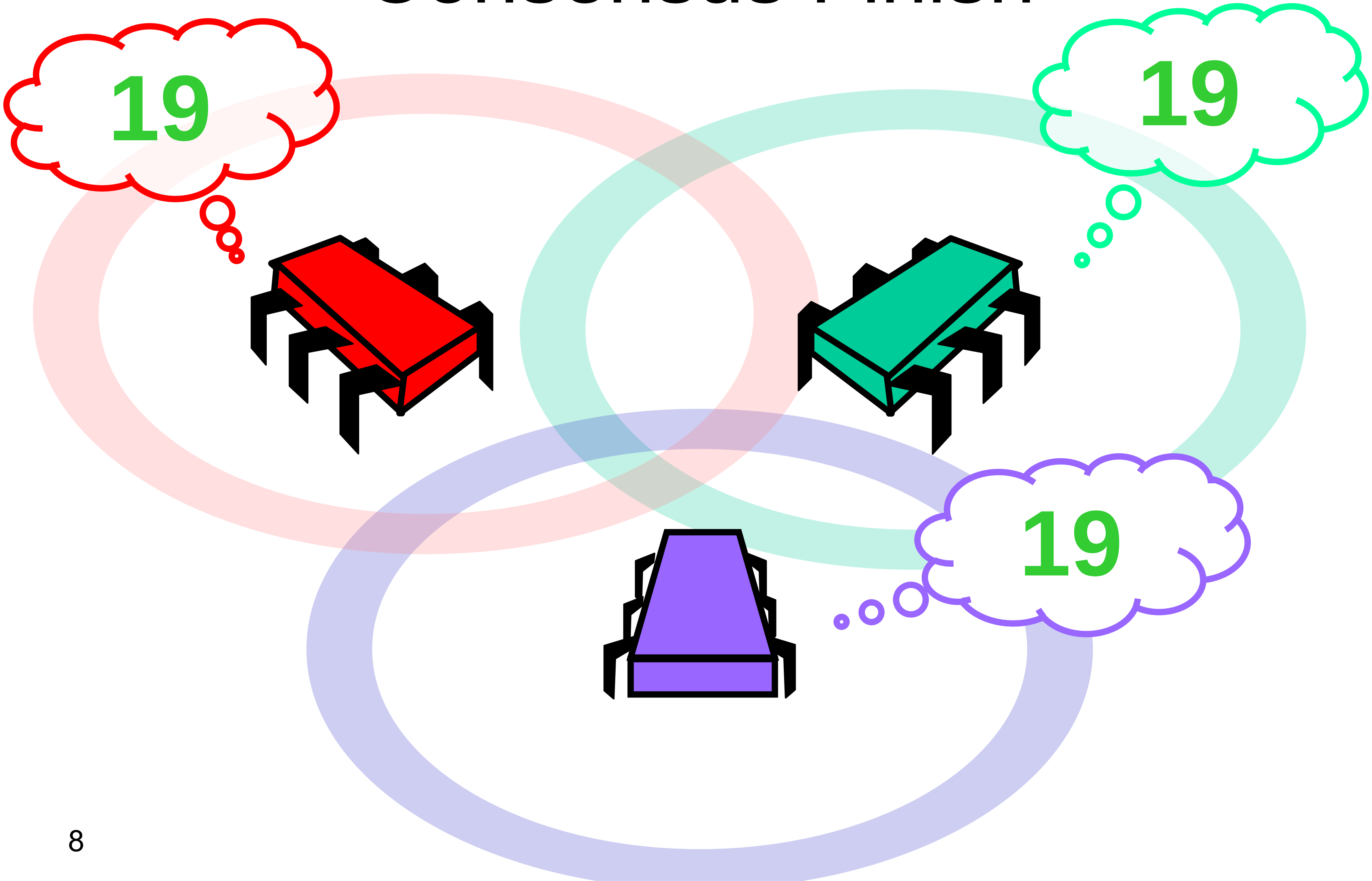
Consensus Start



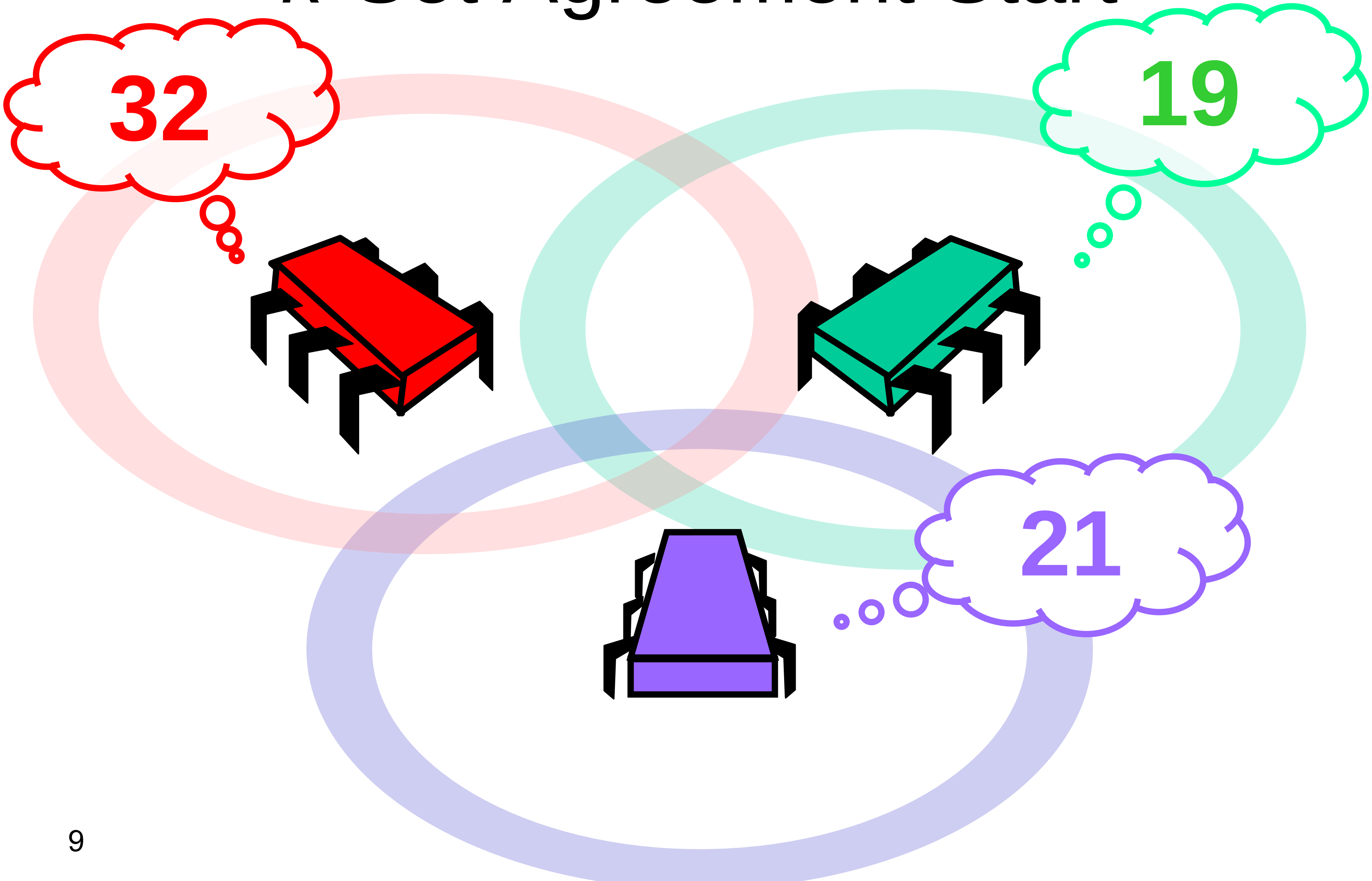
Communication



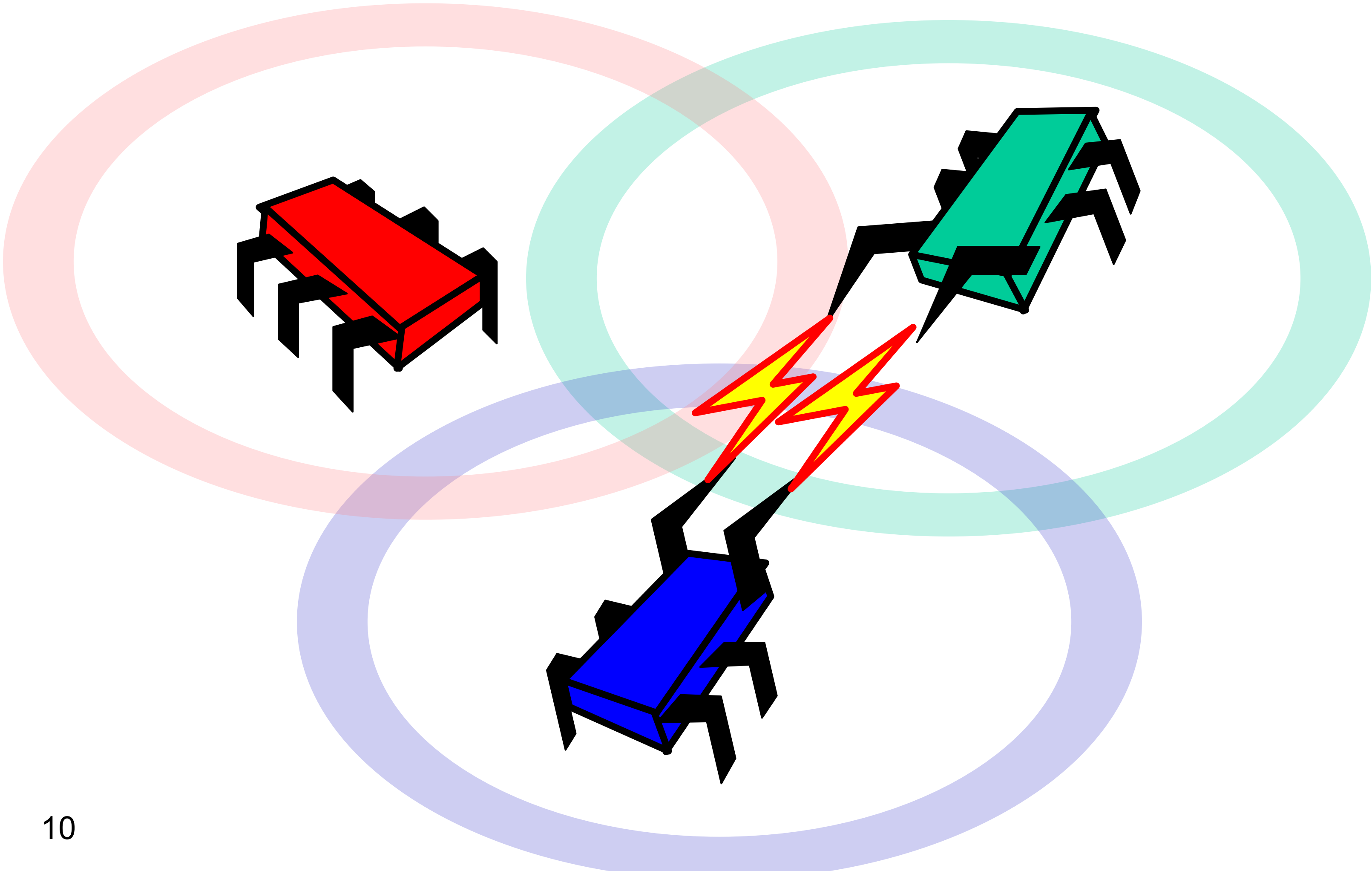
Consensus Finish



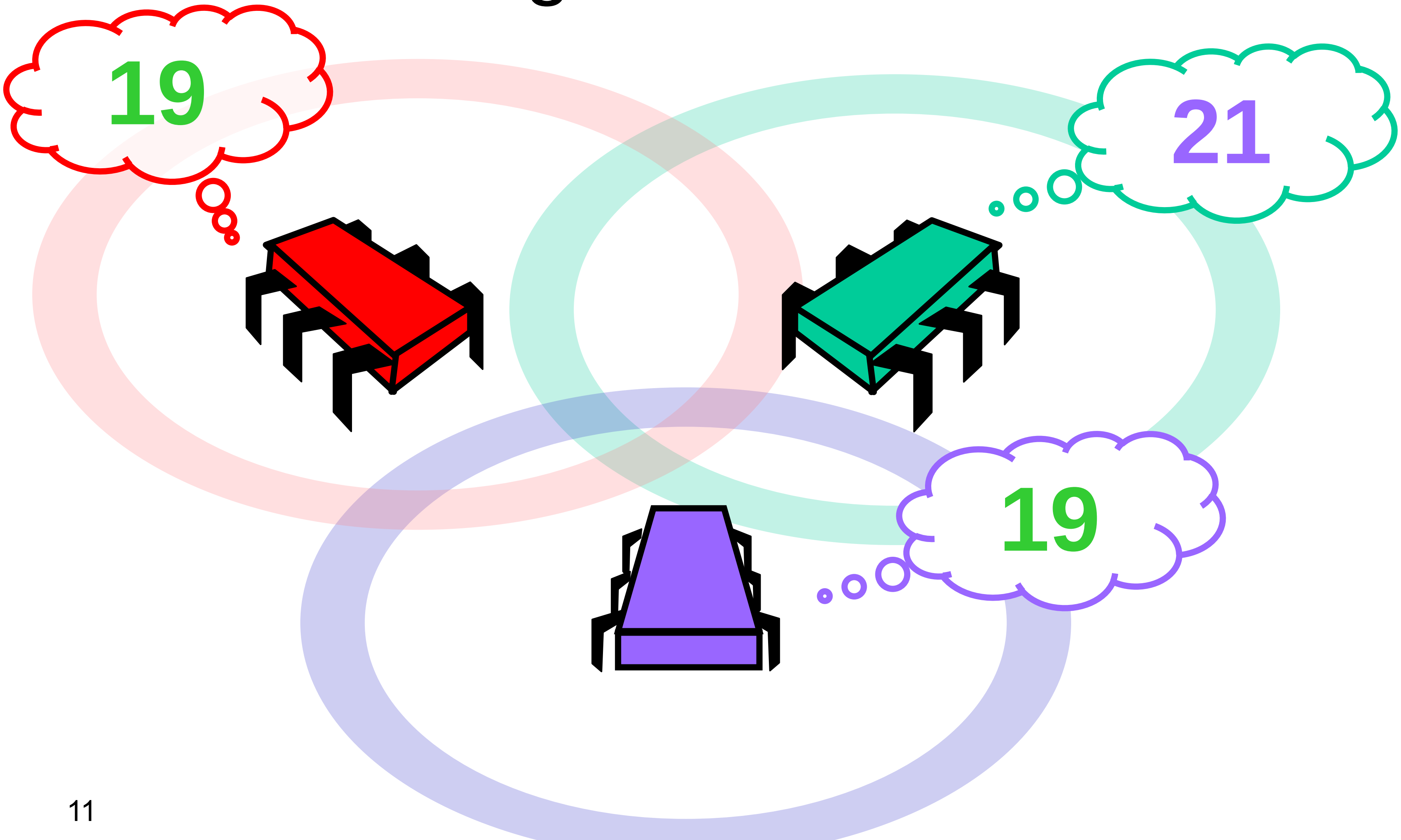
k-Set Agreement Start



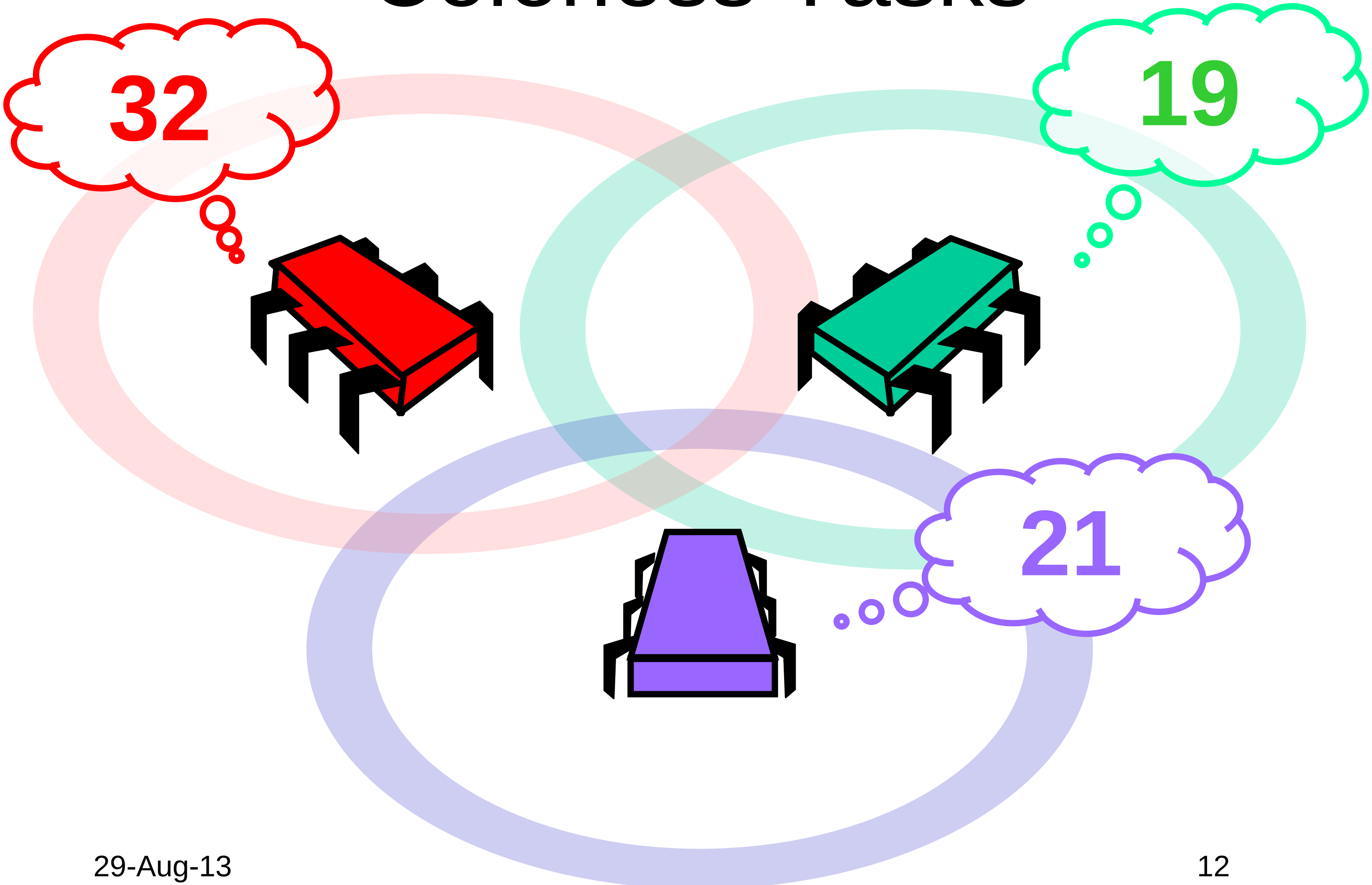
Communication



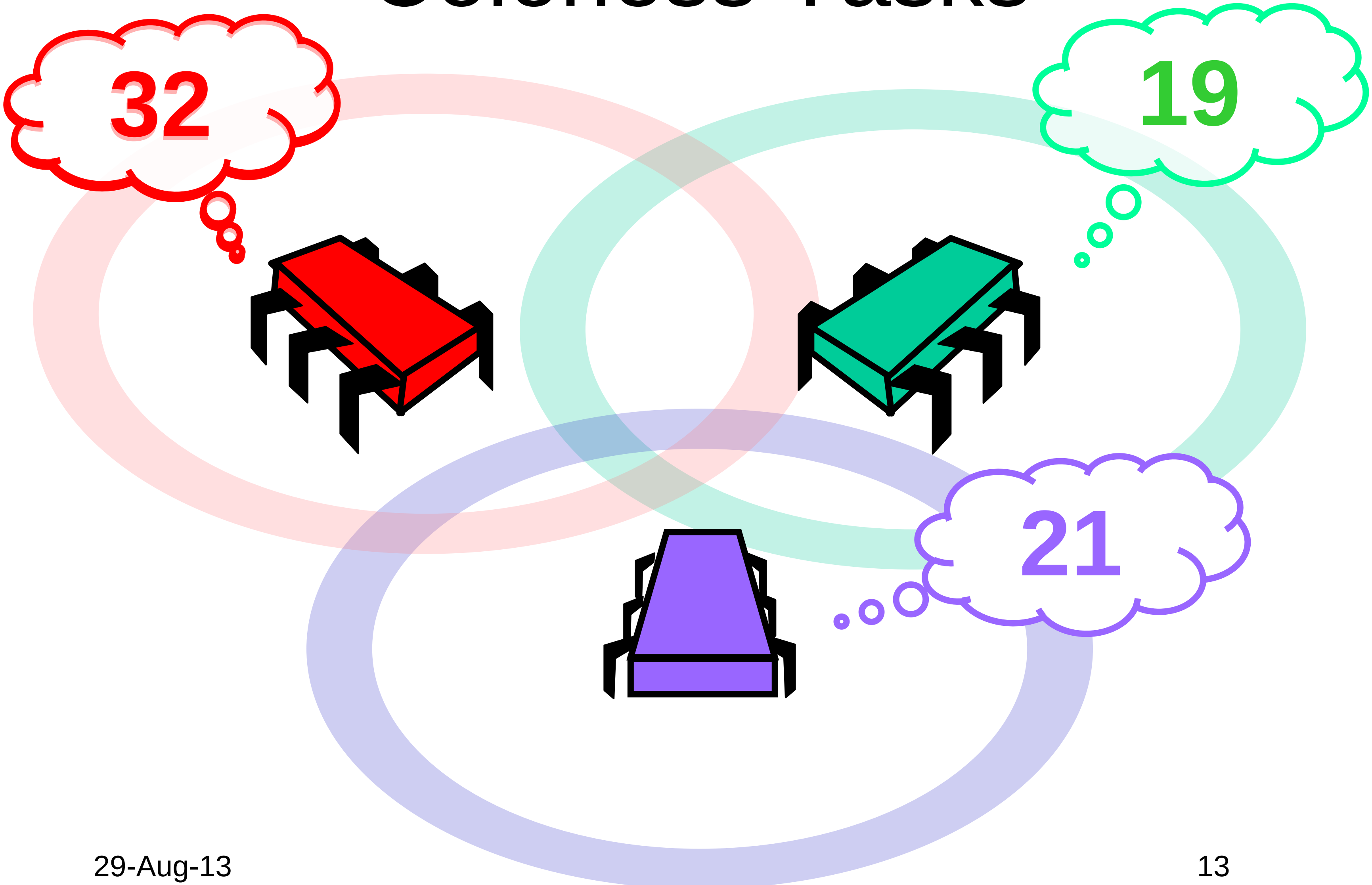
k-Set Agreement Finish



Colorless Tasks



Colorless Tasks



Colorless Tasks

The *set* of input values …

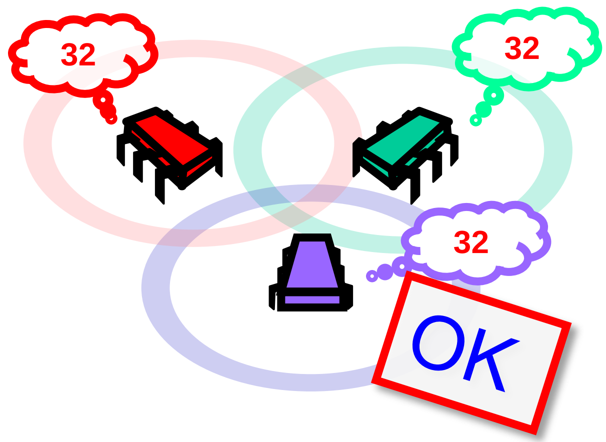
determines the *set* of output values.

Number and identities irrelevant…

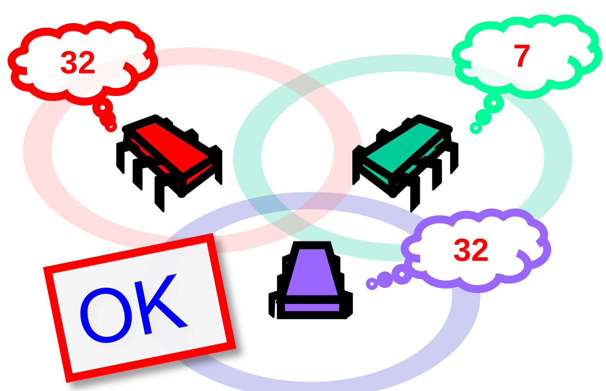
for both input and output values

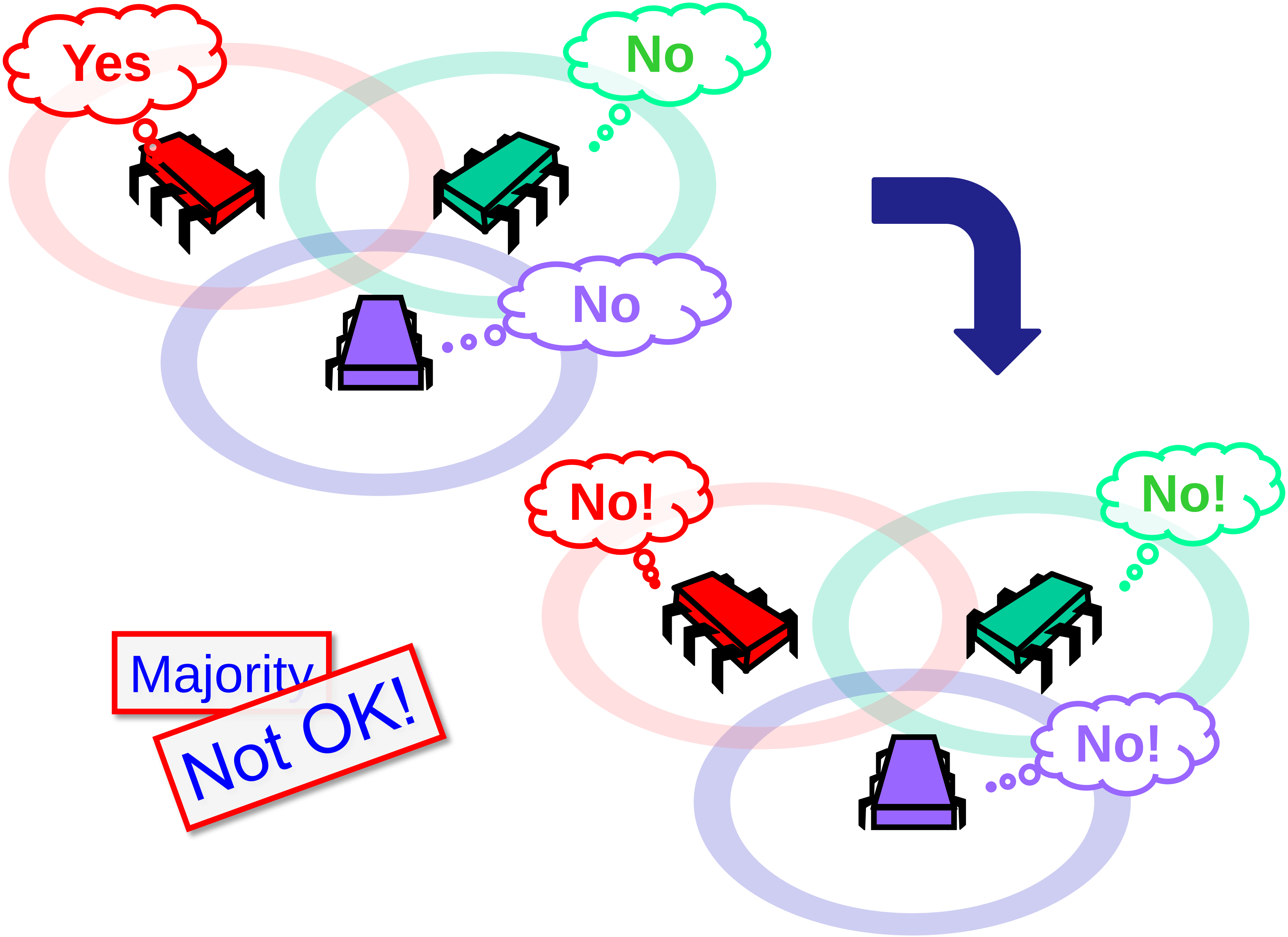
Examples

Consensus



k-set agreement





Yes
No
No
No!
No!
No!
Majority
Not OK!

Road Map

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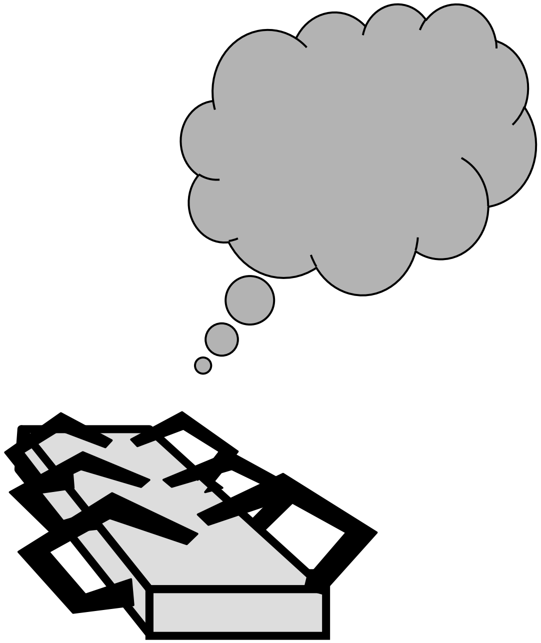
Building Blocks

Crash Failure Solvability

Byzantine Failure Solvability

Failures



Crash failures: processes halt

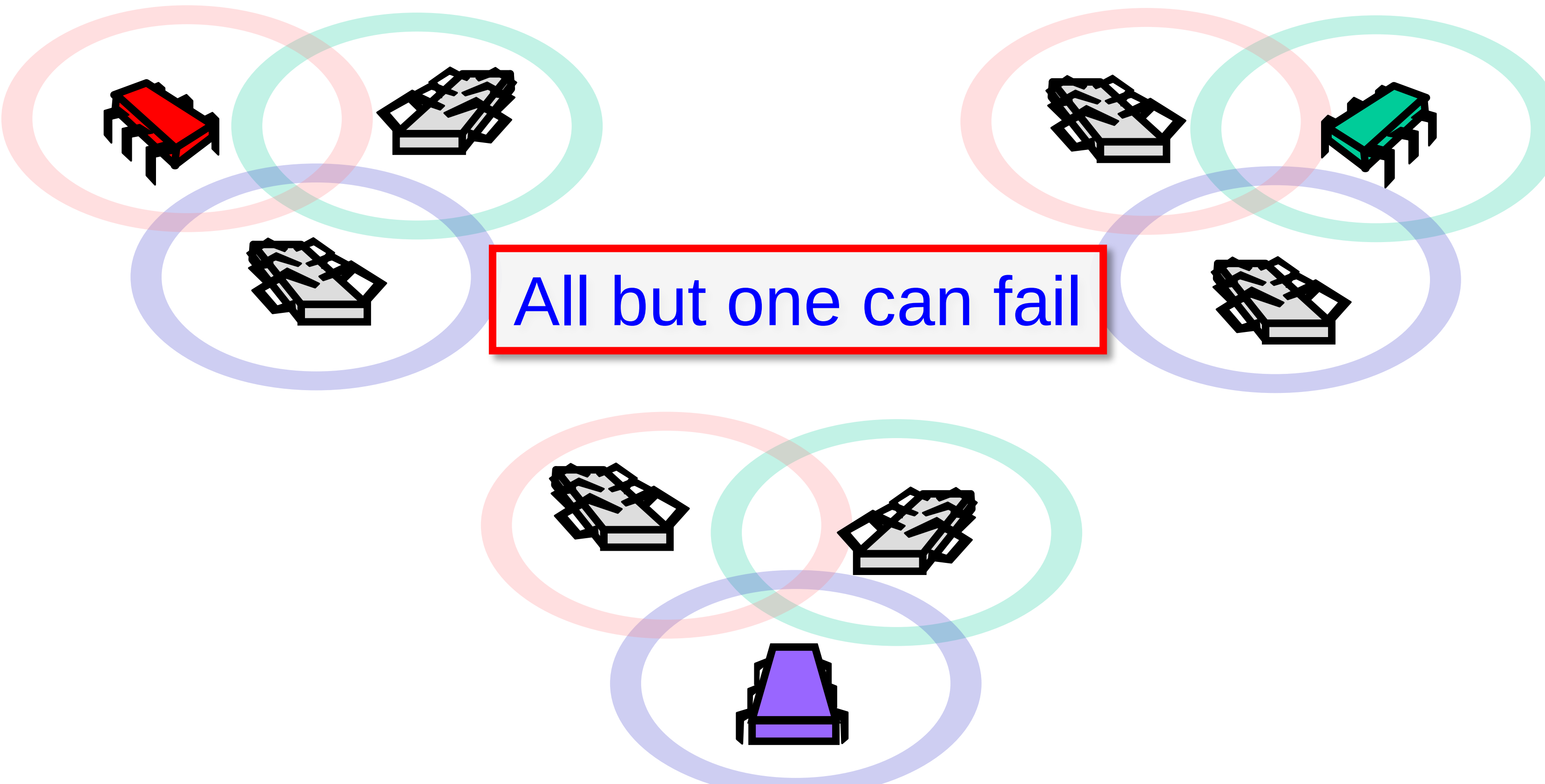
How many?

Failures

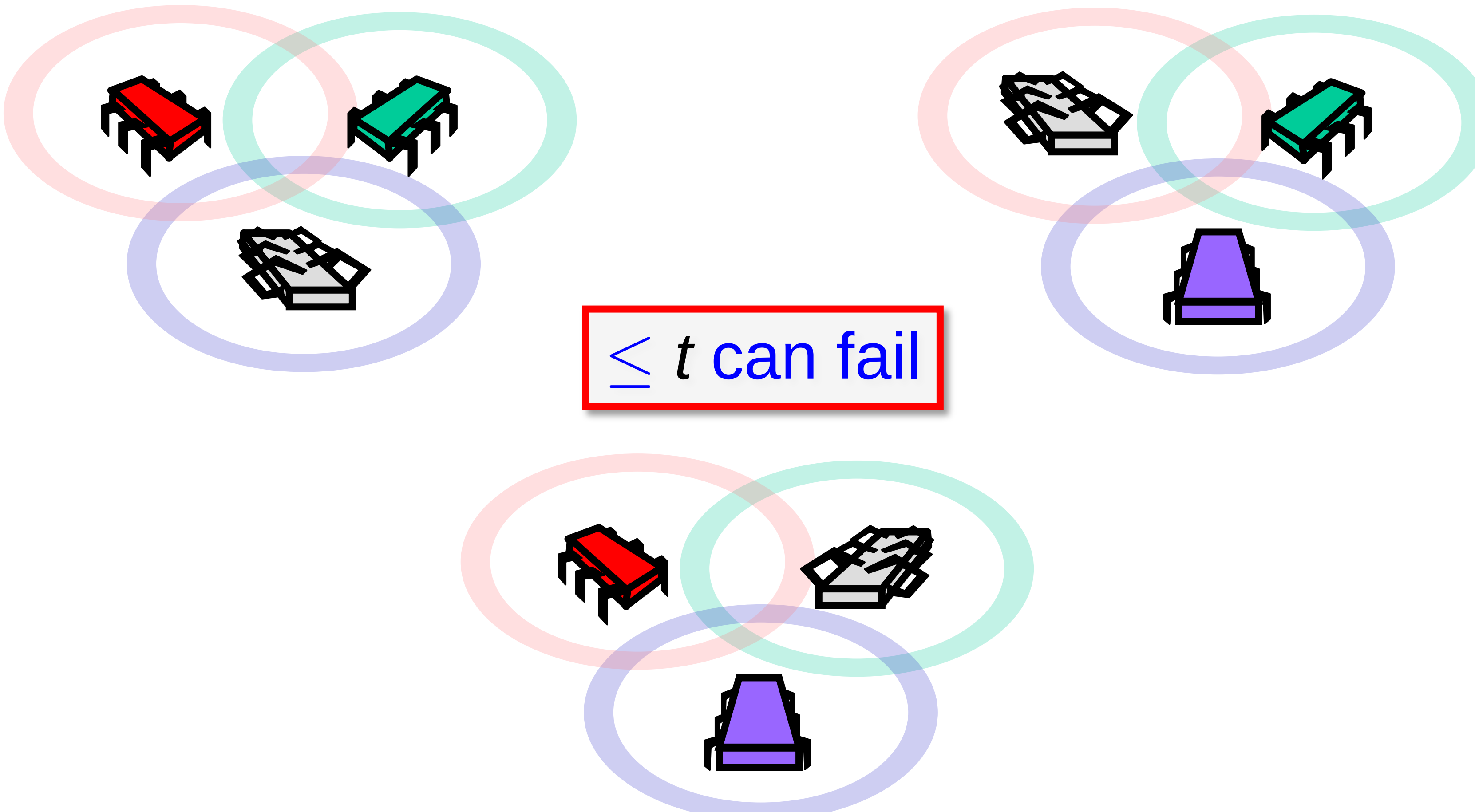
Crash failures: processes misbehave

How many?

Resilience: Wait-Free



Resilience: *t*-resilient



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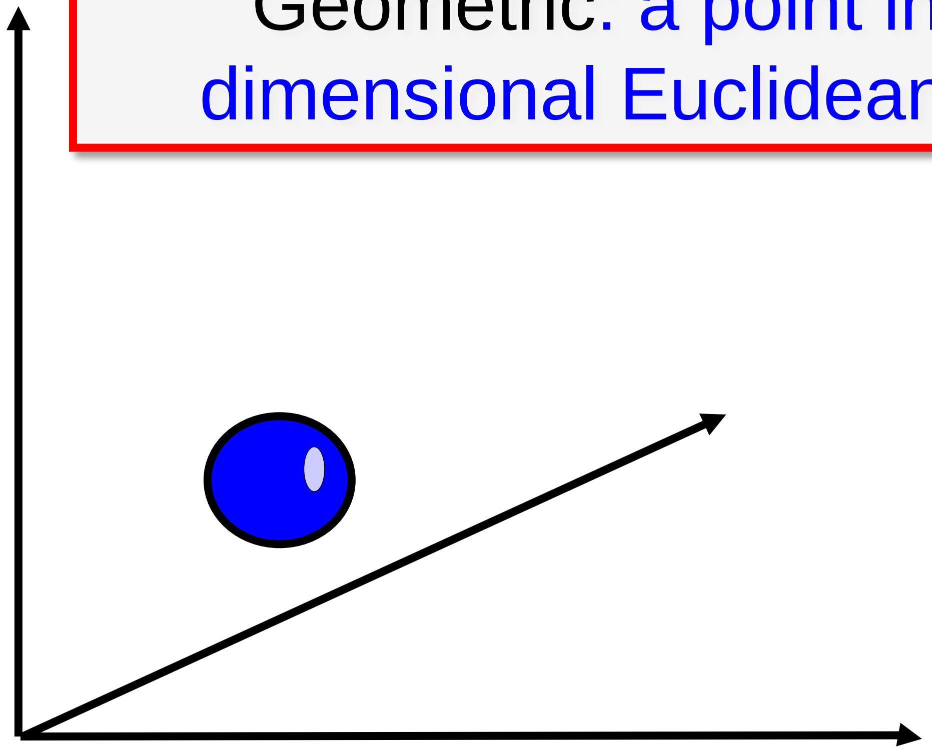
A Vertex

Combinatorial: an element of a set

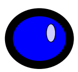
Geometric: a point in high-dimensional Euclidean Space



Simplexes

Combinatorial: a set of vertexes.

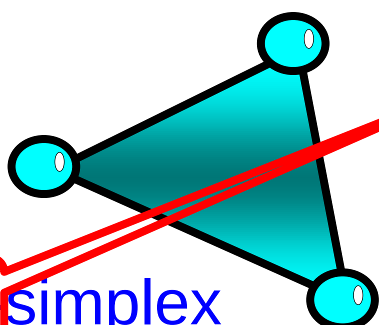
Geometric: convex hull of points in general position



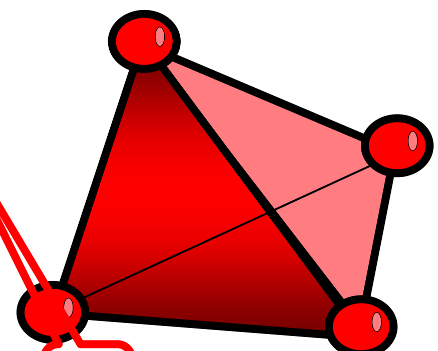
0-simplex

1-simplex

dimension

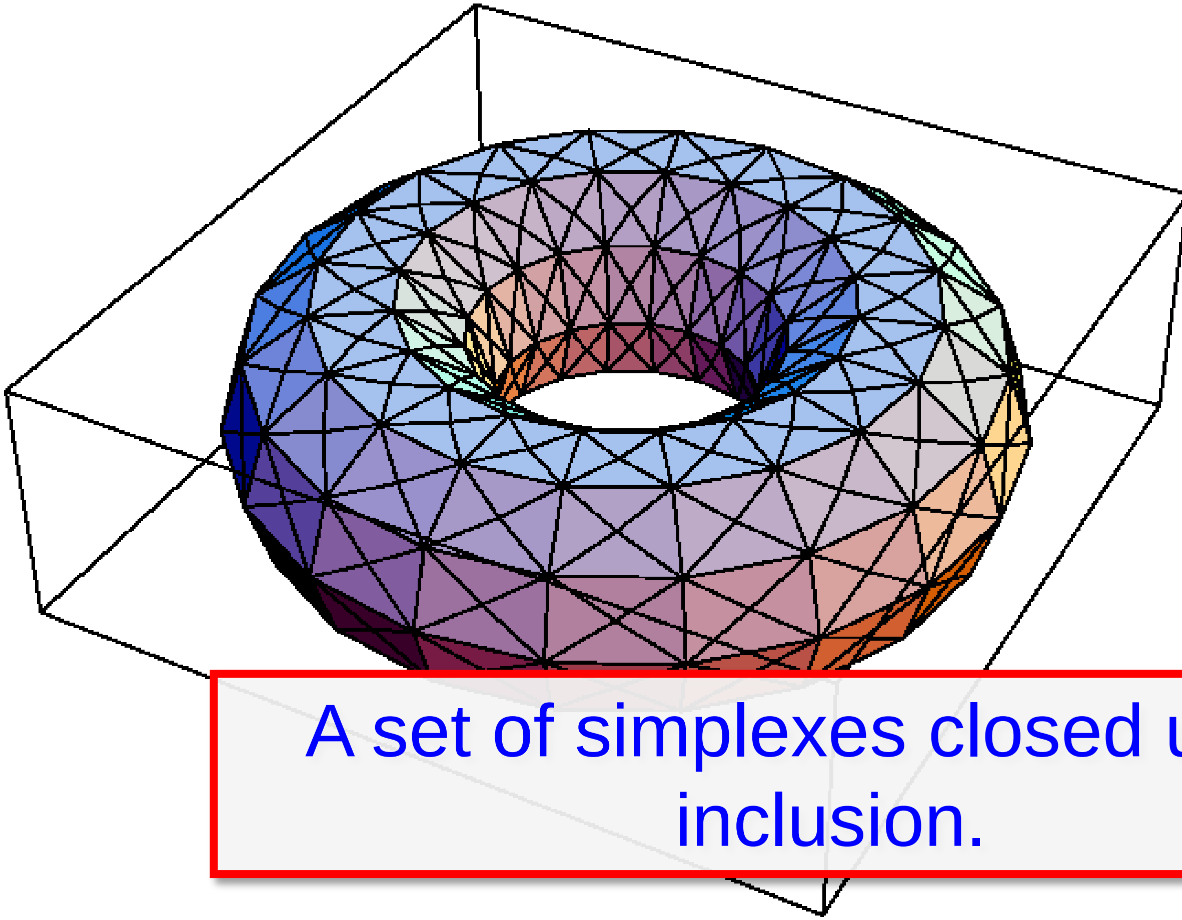


2-simplex

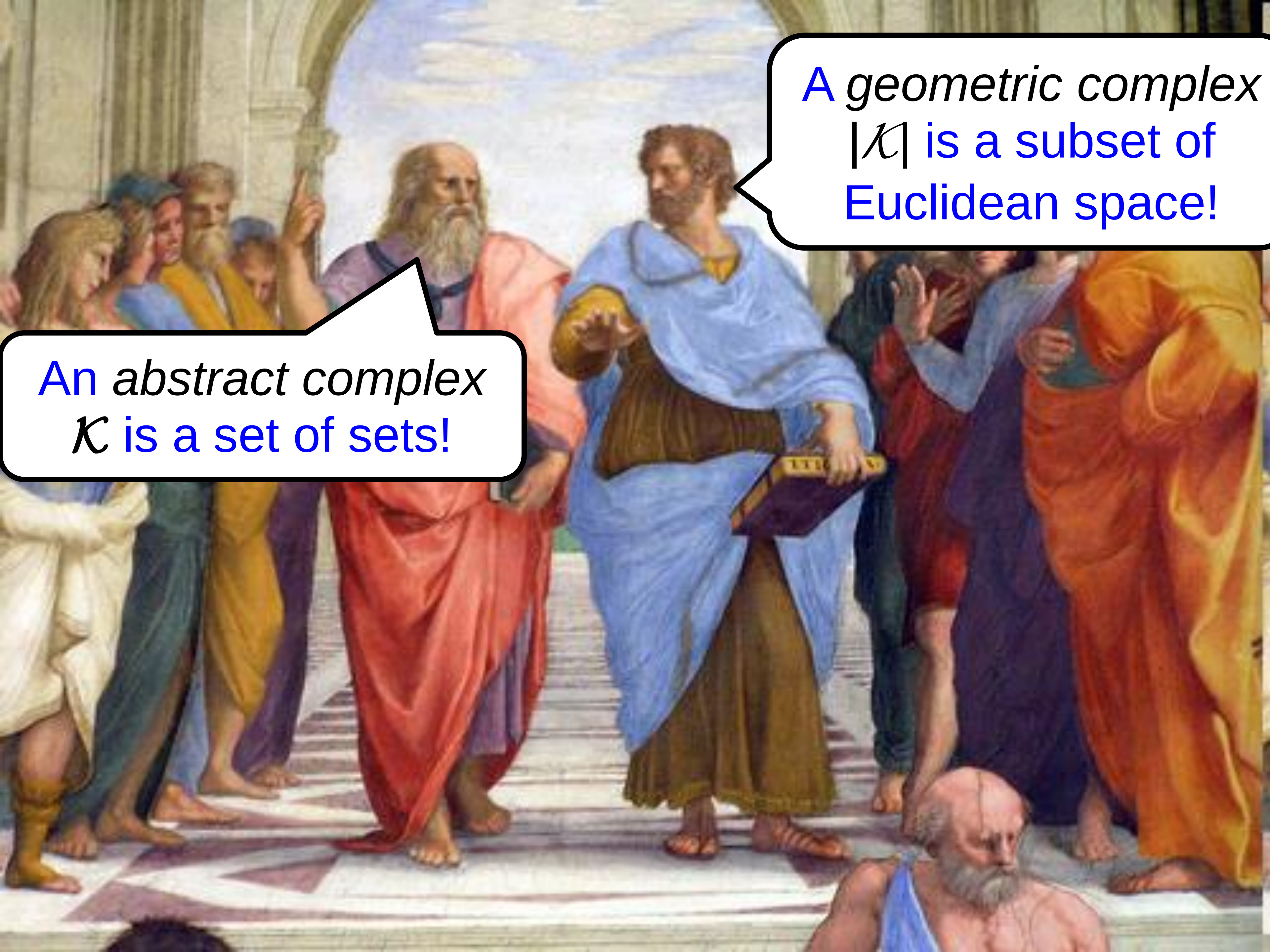


3-simplex

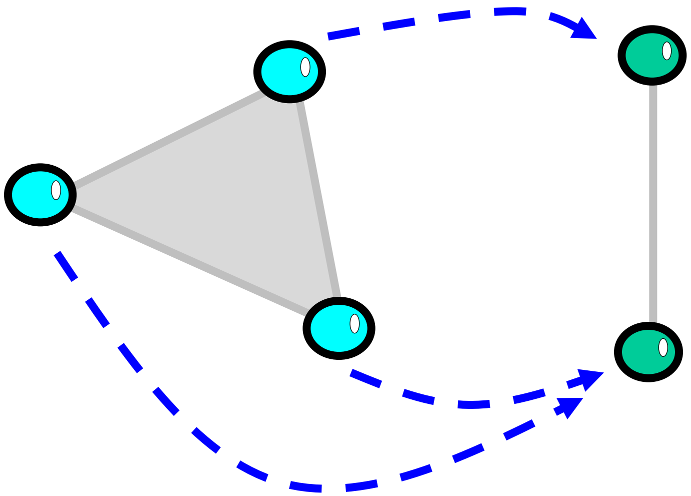
Simplicial Complex



A set of simplexes closed under inclusion.

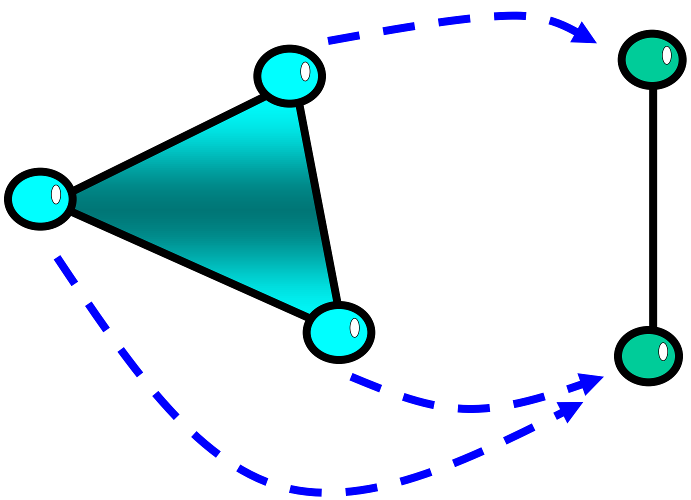

A *geometric complex* $|\mathcal{K}|$ is a subset of Euclidean space!
An *abstract complex* $\mathcal{K}$ is a set of sets!

Simplicial Maps



Vertex-to-vertex map …

Simplicial Map

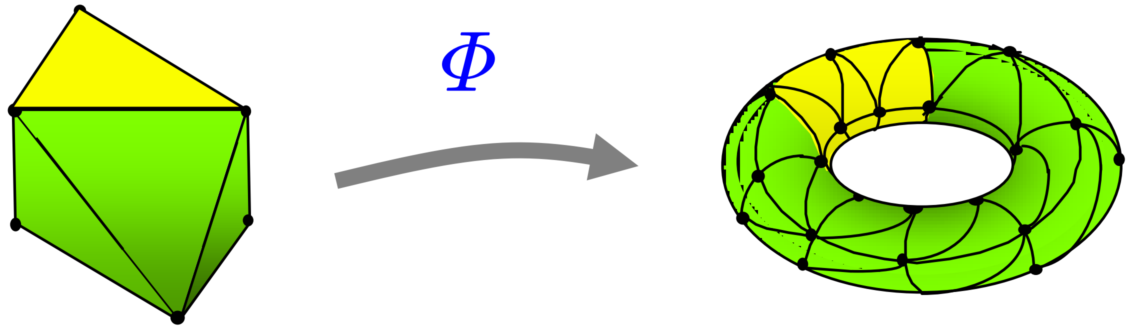


Vertex-to-vertex map …

that sends simplexes to simplexes

piece-wise linear map on geometric simplexes

Carrier Map



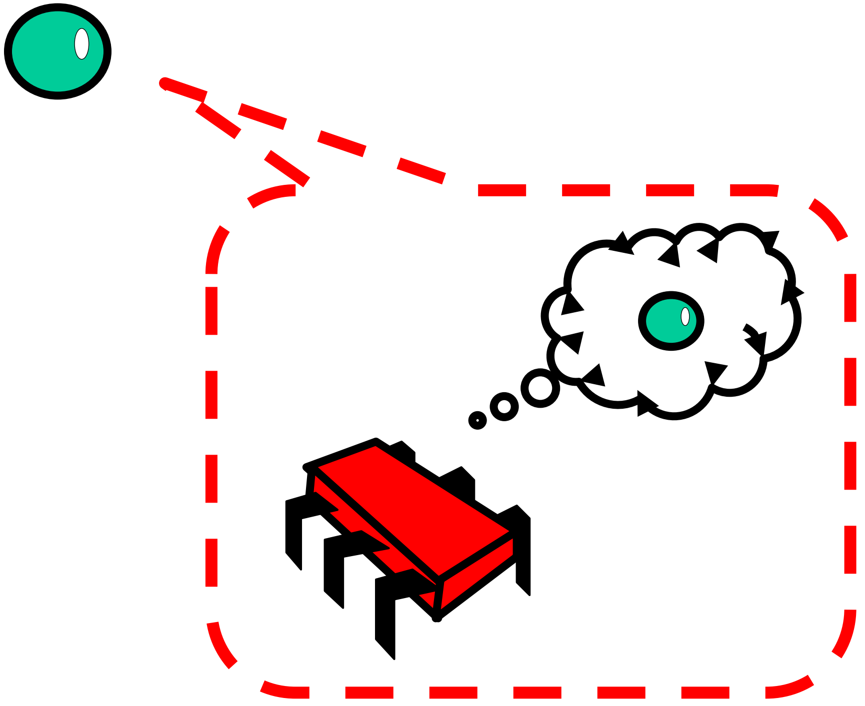
Maps simplex …

to subcomplex.

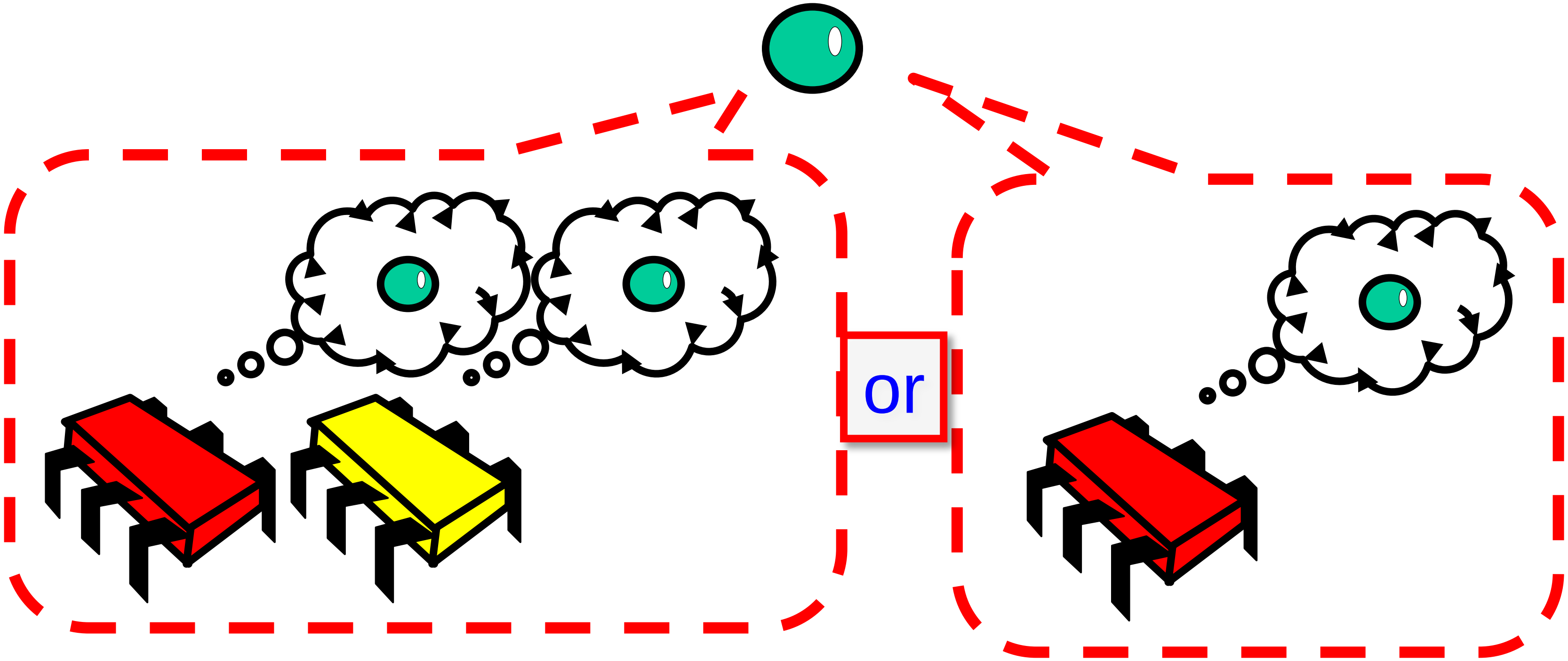
Monotonic: if $\sigma \subset \tau$ then $\Phi(\sigma) \subset \Phi(\tau)$

Always OK to discard inputs

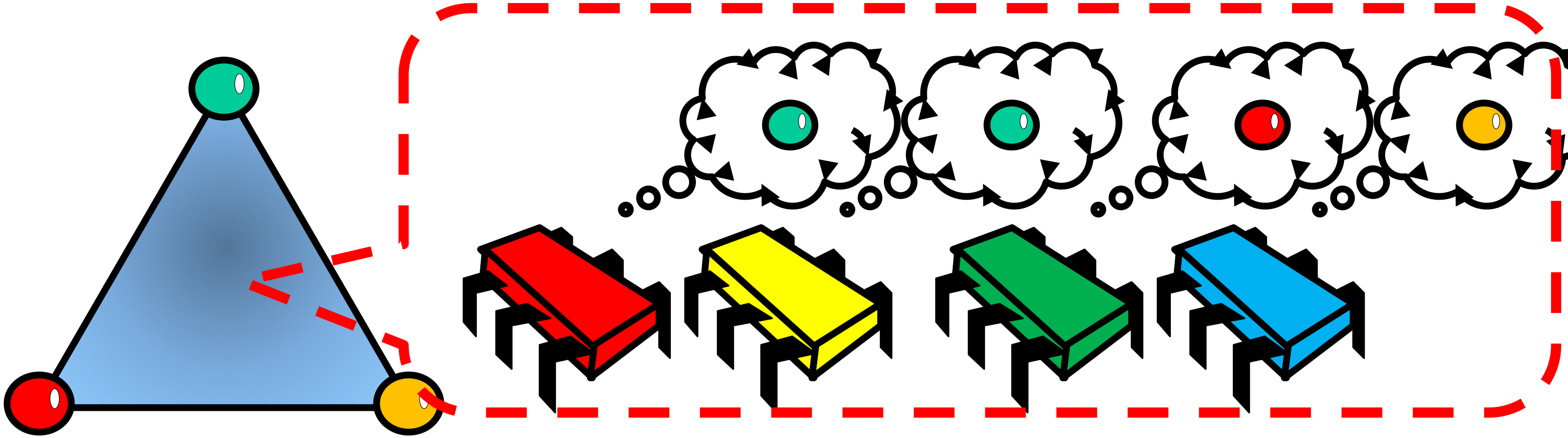
Vertex = Input or Output Value



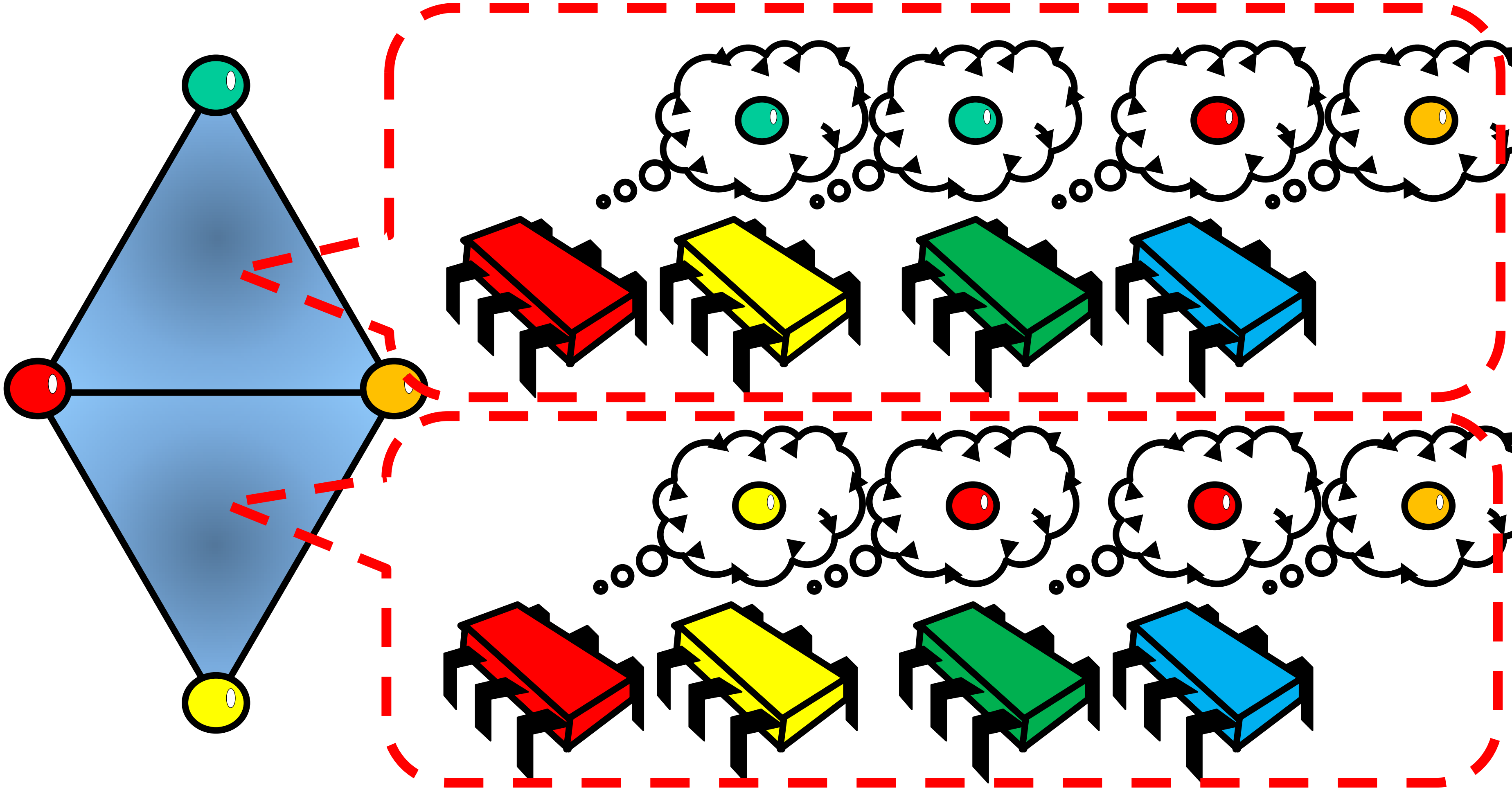
Vertex = Input or Output Value



Simplex = Compatible Values

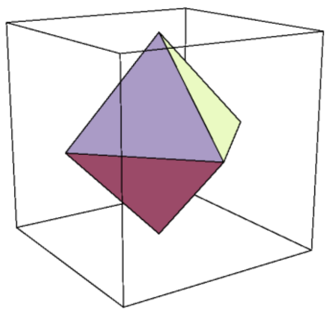


Simplex = Compatible Values

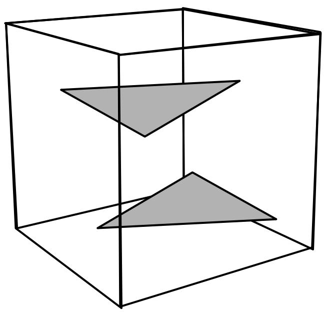


Task Specification

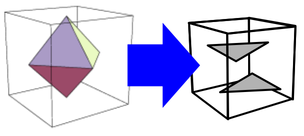
$$(\mathcal{I}, \mathcal{O}, \Delta)$$



Input complex

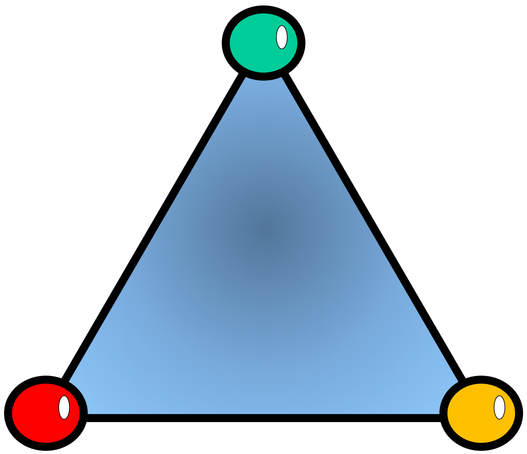


Output complex

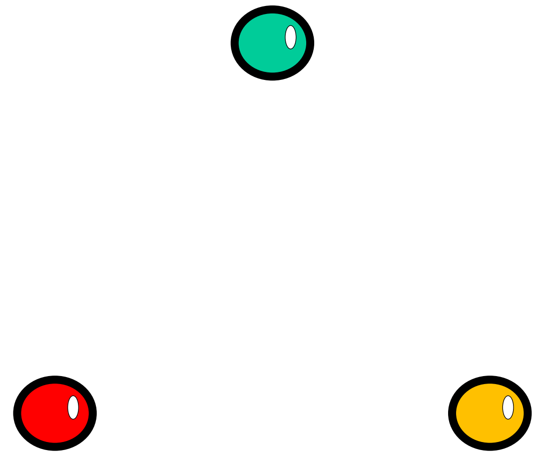


Carrier map

Consensus

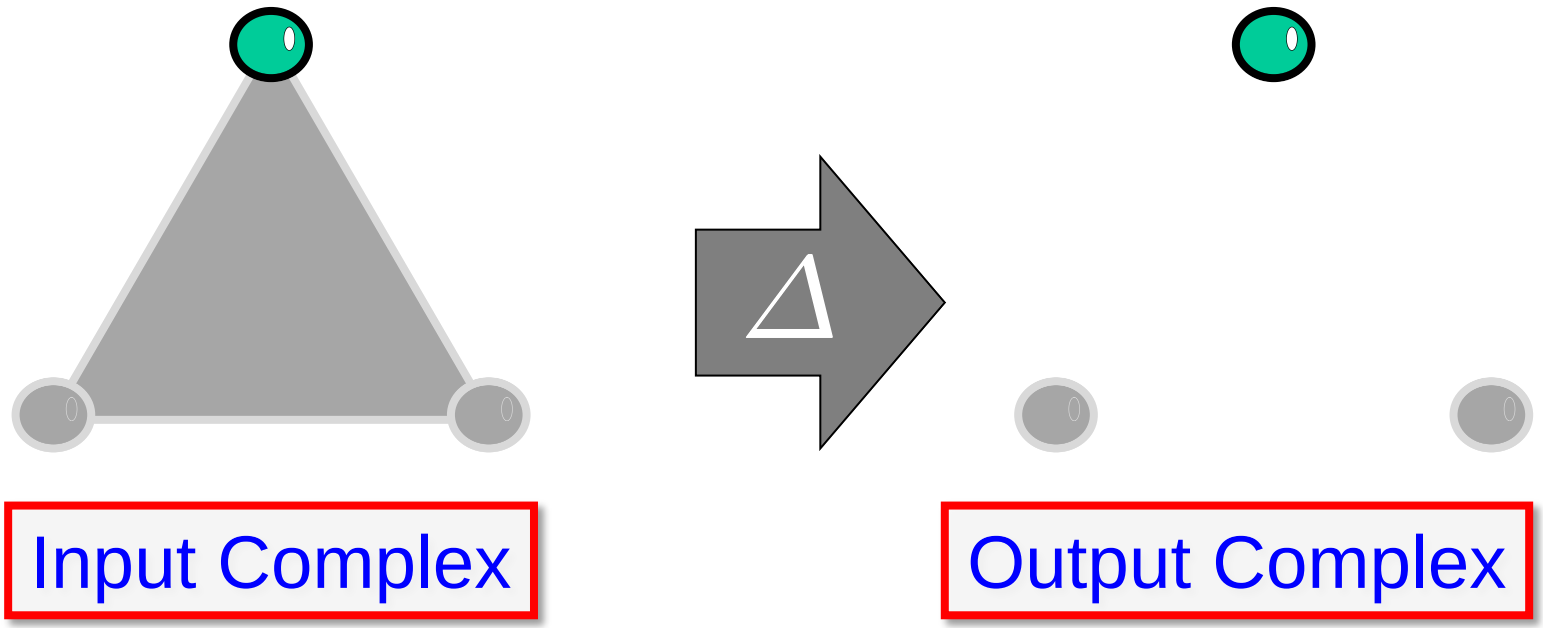


Input Complex

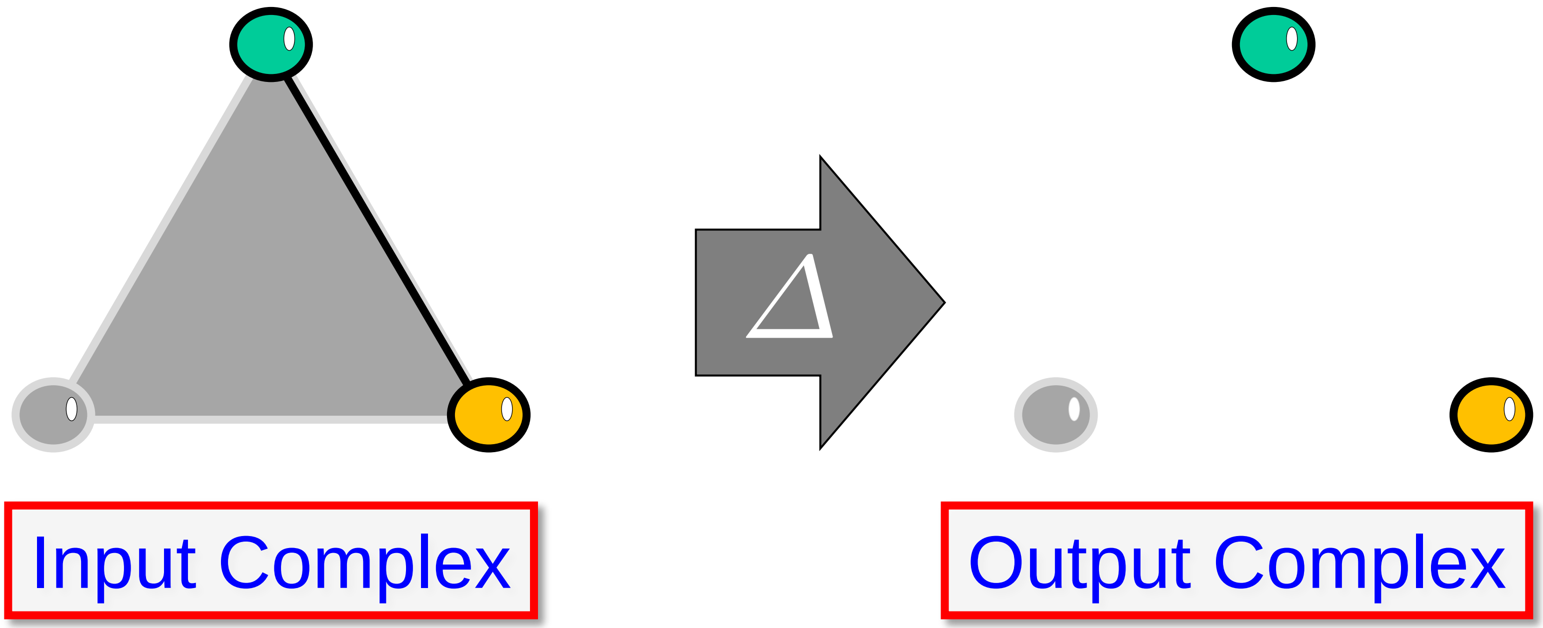


Output Complex

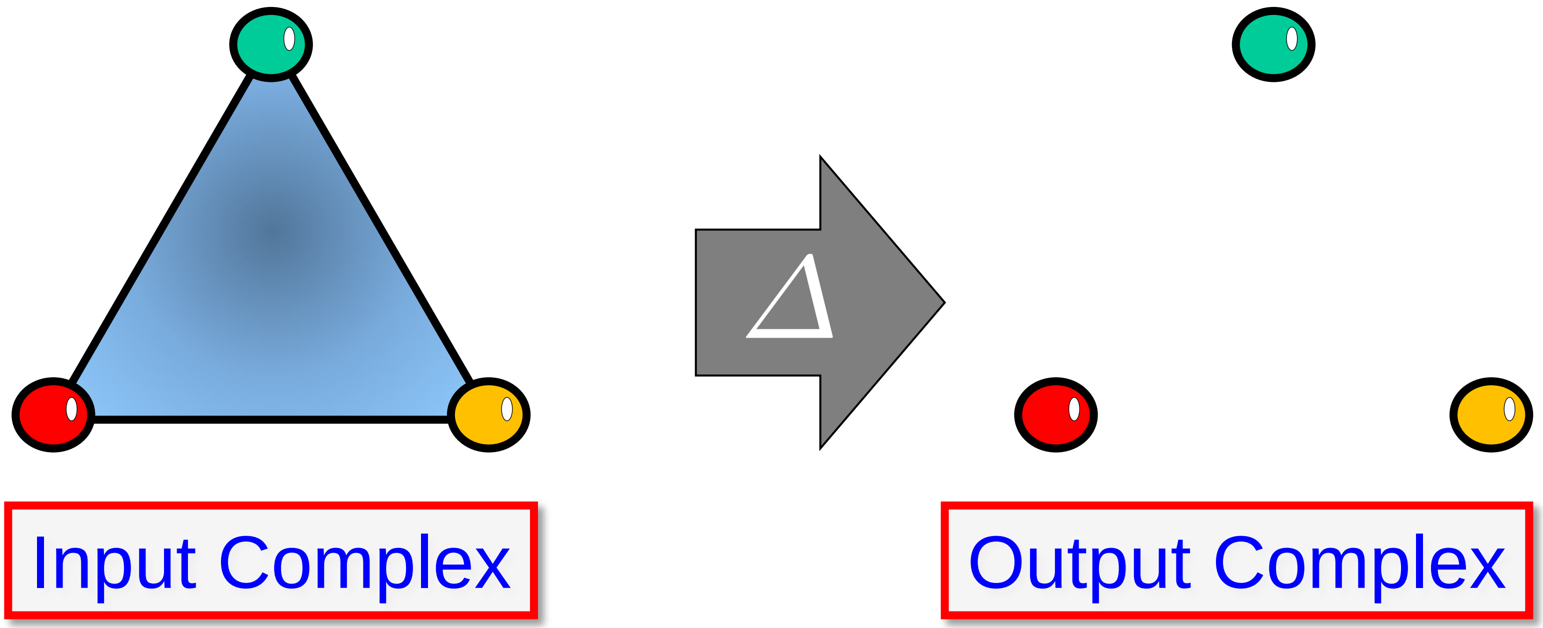
Carrier Map



Consensus



Consensus



Protocol

```
view = my input value;
for (i = 0; i < r; i++) {
    broadcast view;
    view += messages received;
    }
return δ(view)
```

Finite program

Protocol

```
view = my input value;
for (i = 0; i < r; i++) {
    broadcast view;
    view += messages received;
    }
return δ(view)
```

Start with input value

Protocol

```
view = my input value;
for (i = 0; i < r; i++) {
    broadcast view;
    view += messages received;
    }
return δ(view)
```

Run for fixed number of rounds

Protocol

```
view = my input value;
for (i = 0; i < r; i++) {
    broadcast view;
    view += messages received;
    }
return δ(view)
```

Send current view to others

Protocol

```
view = my input value;
for (i = 0; i < r; i++) {
    broadcast view;
    view += messages received;
    }
return δ(view)
```

Concatenate messages received to view
(full-information protocol)

Protocol

```
view = my input value;
for (i = 0; i < r; i++) {
    broadcast view;
    view += messages received;
    }
return δ(view)
```

finally, apply task-specific decision map to view

Protocol Complex

Vertex: possible view

Full information: messages sent & received

Simplex: compatible set of views

Each execution defines a simplex

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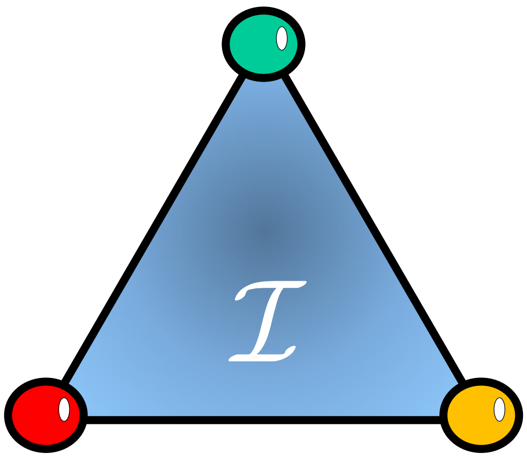
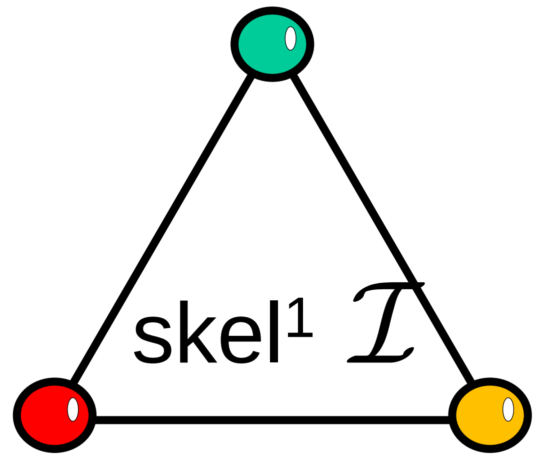
k-Set Agreement



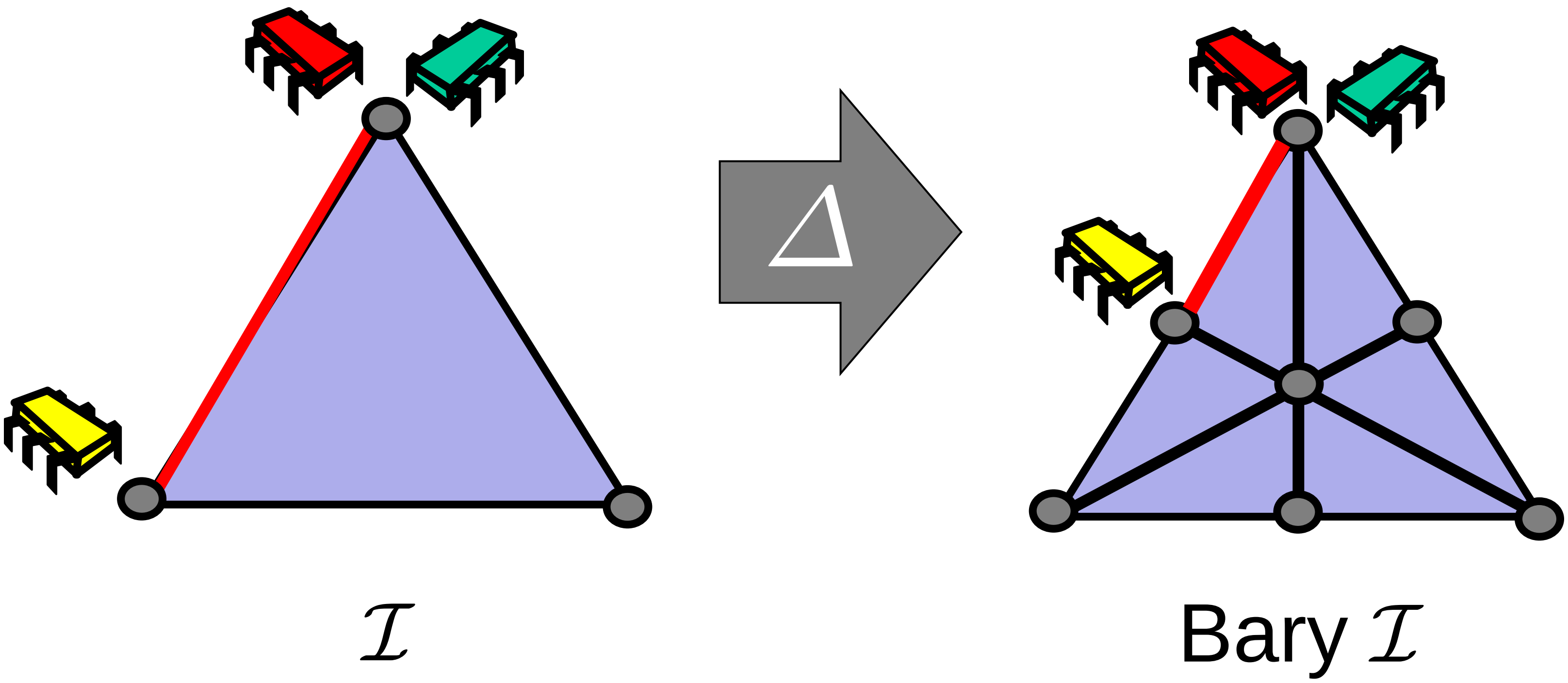
Input Complex



Output Complex

Δ is *k*-skeleton operator

Barycentric Agreement



Δ is *barycentric subdivsion* operator

Closely related to *snapshot*

k-Set Agreement in Crash Failure Model

Theorem

There exists a t-resilient (t+1) -set agreement protocol ($\mathcal{I}$, skel^t $\mathcal{I}$, skel$^t(\cdot)$)

Proof

Broadcast value, wait for all but t values, decide least one.

k-Set Agreement in Byzantine Failure Model

Theorem

There exists a t-resilient $(t+1)$-set agreement protocol

$(\mathcal{I}, \mathrm{skel}^t\, \mathcal{I}, \mathrm{skel}^t(\cdot))$

Iff $n+1 > t \cdot (\dim \mathcal{I} + 2)$

new!

Cost of Byzantine failures!

The Necessary Part

Byzantine processes cannot influence decisions!

Non-Faulty processes cannot “believe” value with < t+1 witnesses

If $n+1 > t \cdot (\dim \mathcal{I} + 2)$ then some value has at least t+1 witnesses

The Sufficient Part

Variation of *reliable broadcast* protocol of [Bracha 87] and [Shrikanth &Toueg 87]

Non-Faulty processes agree on values sent by others, even faulty processes.

If one non-faulty process receives a message, so do the others (liveness)

Barycentric Agreement in Crash Failure Model

Theorem

There is a *t*-resilient barycentric agreement protocol
$(\mathcal{I}, \mathrm{Bary}^N \mathcal{I}, \mathrm{Bary}^N(\cdot))$

Proof

Variation of *stable vectors* algorithm of
[Attiya et al. 90]

Barycentric Agreement in Byzantine Failure Model

Theorem

There is a *t*-resilient barycentric agreement protocol

$(\mathcal{I}, \mathrm{bary}\mathcal{I}^t\, \mathcal{I}, \mathrm{bary}\mathcal{I}^t(\cdot))$

Iff $n+1 > t \cdot (\dim \mathcal{I} + 2)$

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Byzantine processes cannot influence decisions!

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If $n+1 > t \cdot (\dim \mathcal{I} + 2)$ then some value has at least t+1 witnesses

The Sufficient Part

Byzantine variation of *stable vectors* algorithm of [Attiya et al. 90]

Use reliable broadcast to spread values

Ignore values with fewer than t+1 witnesses …

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Solvability for Crash Failures

Theorem

There is a t-resilient protocol for task

$(\mathcal{I}, \mathcal{O}, \Delta)$

Iff there is a continuous map

$f\colon |\mathrm{skel}^t\, \mathcal{I}| \to |\mathcal{O}|$

carried by Δ

$f(|\sigma|) \subseteq \Delta(\sigma)$

The Necessary Part

We can model the set of process views after a full-information protocol as a *protocol complex* $\mathcal{P}$

There is a simplicial decision map

δ: $\mathcal{P} \to \mathcal{O}$ carried by Δ

WLOG, $\mathcal{P}$ is isomorphic to $\text{bary}^N \text{skel}^t \mathcal{I}$

δ: $\text{bary}^N \text{skel}^t \mathcal{I} \to \mathcal{O}$

induces a piece-wise linear map

$|\delta|: |\text{bary}^N \text{skel}^t \mathcal{I}| \to |\mathcal{O}|$

$|\delta|: |\text{skel}^t \mathcal{I}| \to |\mathcal{O}|$

carried by Δ

The Sufficient Part

f: $|\text{skel}^t\, \mathcal{I}| \to |\mathcal{O}|$
has a *simplicial approximation* for some N > 0
ϕ: $\text{bary}^N\, \text{skel}^t\, \mathcal{I} \to \mathcal{O}$

Step 1: use *t*-set agreement protocol to go from vertex of $\mathcal{I}$ to vertex of $\text{skel}^t\, \mathcal{I}$

Step 2: use repeated barycentric agreement to go from vertex of $\text{skel}^t\, \mathcal{I}$ to vertex of : $\text{bary}^N\, \text{skel}^t\, \mathcal{I}$

Step 3: from vertex $v \in$: $\text{bary}^N\, \text{skel}^t\, \mathcal{I}$, decide $\phi(v)$

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Solvability for Byzantine Failures

Theorem

There is a t-resilient protocol for task $(\mathcal{I}, \mathcal{O}, \Delta)$

Iff there is a continuous map

$f\colon |\mathrm{skel}^t\, \mathcal{I}| \to |\mathcal{O}|$

~~carried by Δ~~

Iff $n+1 > t \cdot (\dim \mathcal{I} + 2)$

new!

Cost of Byzantine failures!

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We can model the set of process views after a full-information protocol as a *protocol complex* $\mathcal{P}$

There is a simplicial decision map
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$|\delta|: |\text{bary}^N \text{skel}^t \mathcal{I}| \to |\mathcal{O}|$

$|\delta|: |\text{skel}^t \mathcal{I}| \to |\mathcal{O}|$

carried by Δ

The Necessary Part

We can model the set ... f process views after a full-information pro... ...s a *protocol complex* $\mathcal{P}$

Ther... ...mplicial decision map
...δ: $\mathcal{P} \to \mathcal{O}$ carried by Δ

...OG, $\mathcal{P}$ is is...rphic to skel^t bary^N $\mathcal{I}$

δ: ... $\mathcal{I} \to \mathcal{O}$
induces a piece-wi...
$|\delta|$: $|\text{skel}^t \text{ bary}^N \mathcal{I}| \to |\mathcal{O}|$
$|\delta|$: $|\text{skel}^t \mathcal{I}| \to |\mathcal{O}|$
carried by Δ

Basic Idea is the same

Technical details more involved

The Sufficient Part

f: $|\mathrm{skel}^t \mathcal{I}| \to |\mathcal{O}|$
has a *simplicial approximation* for some N > 0
ϕ: $\mathrm{bary}^N\ \mathrm{skel}^t\ \mathcal{I} \to \mathcal{O}$

Step 1: use *t*-set agreement protocol to go from vertex of $\mathcal{I}$ to vertex of $\mathrm{skel}^t\ \mathcal{I}$

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Step 3: from vertex $v \in$: $\mathrm{bary}^N\ \mathrm{skel}^t\ \mathcal{I}$, decide $\phi(v)$

The Sufficient Part

f: $|\text{skel}^t$ ... $| \to |\mathcal{O}|$
has a *simplicial* ... *ation* for some N > 0
... $\text{bary}^N \mathcal{I} \to \mathcal{O}$

S... use t-set agreement protocol to
go from vert... of $\mathcal{I}$ to vertex of $\text{skel}^t \mathcal{I}$

Step 2: use repea... agreement
to go from vertex of ...
to vertex of : $\text{skel}^t\ \text{bary}^N \mathcal{I}$

Step 3: from vertex $v \in$: $\text{skel}^t\ \text{bary}^N \mathcal{I}$,
decide $\phi(v)$

Basic Idea is the same

Technical details more involved

Conclusions

Many have looked at specific tasks …

consensus

approximate agreement

k-set agreement

Often with much weaker validity!

First to look at general (colorless) tasks …

First to characterize what can and can't be solved

Conclusions

The *language* of combinatorial topology (*vertex, simplex, skeleton, simplicial map* …) allows us to *state* and *prove* such results succinctly

Important to exploit the duality of combinatorial and continuous model (such as *simplicial approximation*)

Here, we did not need “advanced” concepts like connectivity, but they are needed elsewhere, such as the synchronous model …

Open Problems

Colored tasks?

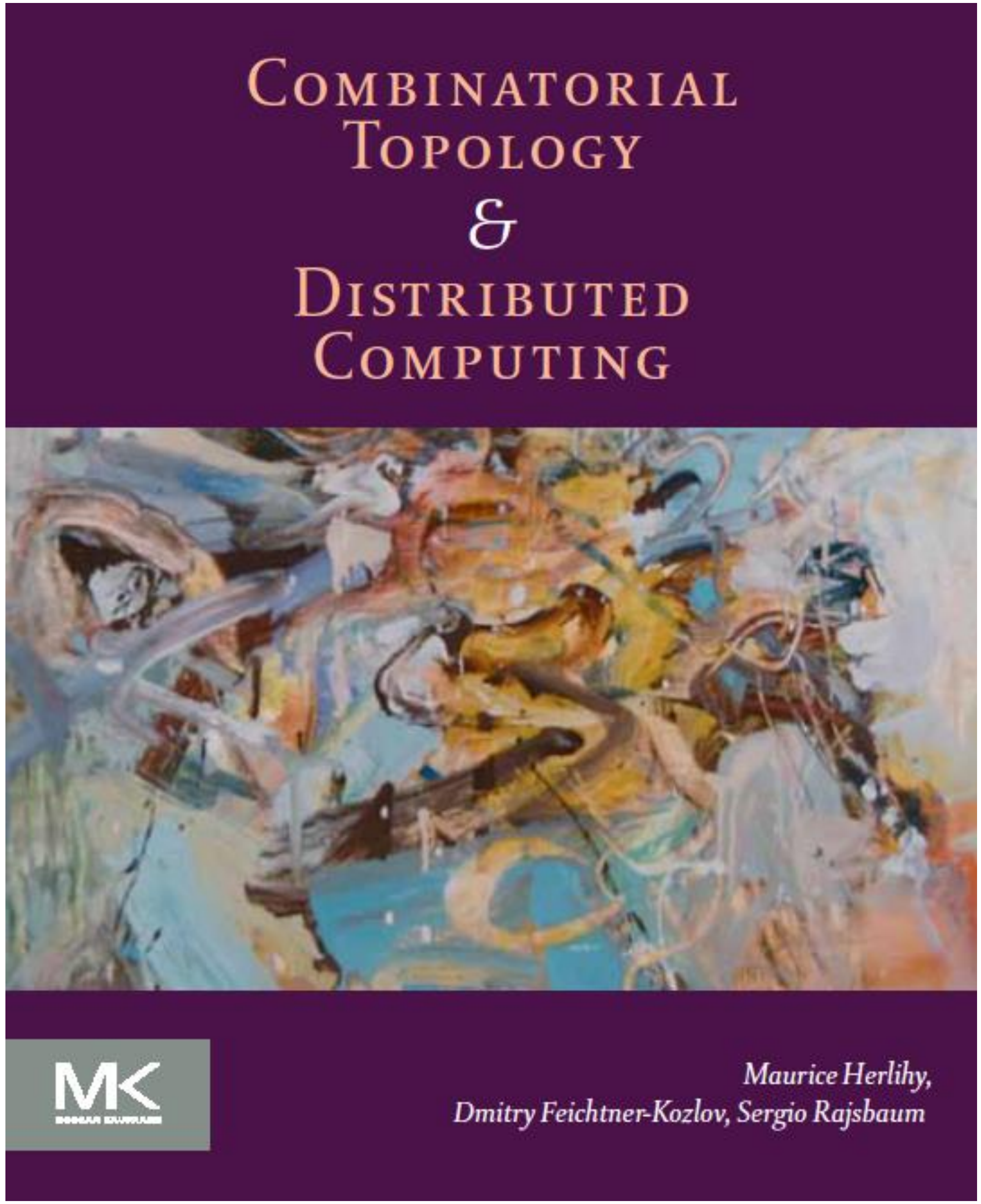
Complexity?

"Rational" adversaries?

Mechanism design?

Long-lived computations?

Randomized?


COMBINATORIAL TOPOLOGY
&
DISTRIBUTED COMPUTING
Maurice Herlihy,
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