Tensor theta norms

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Overview

Compressive sensing - vector case

- $\mathbf{\Phi} \in \mathbb{R}^{m \times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x}$ sparse, $\mathbf{b} = \mathbf{\Phi x} \in \mathbb{R}^m$, $m \ll n$
- Goal: Reconstruct $\mathbf{x}$ from $\mathbf{\Phi}$ and $\mathbf{b}$
- ℓ_0-minimization (non-convex optimization problem)

$$\min_{\mathbf{z} \in \mathbb{R}^n} \|\mathbf{z}\|_0 \text{ such that } \mathbf{\Phi z} = \mathbf{b} \qquad \text{(NP-hard)}$$

  where

$$\|\mathbf{x}\|_0 = \#\{i : |x_i| \neq 0\}.$$

- ℓ_1-minimization (convex optimization problem)

$$\min_{\mathbf{z} \in \mathbb{R}^n} \|\mathbf{z}\|_1 \text{ such that } \mathbf{\Phi z} = \mathbf{b}.$$

- Gaussian matrix: with high probability all s-sparse vectors can be reconstructed provided $m \geq Cs \ln(n/s)$.

Compressive sensing - matrix case

- $\mathbf{\Phi} : \mathbb{R}^{n_1 \times n_2} \to \mathbb{R}^m$, $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$, $\mathbf{b} = \mathbf{\Phi}(\mathbf{X}) \in \mathbb{R}^m$, $\mathbf{X}$ is of low rank, $m \ll n_1 n_2$
- Goal: Reconstruct $\mathbf{X}$ from $\mathbf{\Phi}$ and $\mathbf{b}$
- non-convex optimization problem

$$\min_{\mathbf{Z} \in \mathbb{R}^{n_1 \times n_2}} \operatorname{rank}(\mathbf{Z}) \text{ such that } \mathbf{\Phi}(\mathbf{Z}) = \mathbf{b}. \qquad \text{(NP-hard)}$$

- nuclear norm minimization (convex optimization problem)

$$\min_{\mathbf{Z} \in \mathbb{R}^{n_1 \times n_2}} \|\mathbf{Z}\|_* \text{ such that } \mathbf{\Phi}(\mathbf{Z}) = \mathbf{b},$$

where $\|\mathbf{Z}\|_* = \sum_{i=1}^{r} \sigma_i = \|\sigma(\mathbf{Z})\|_1$, σ_i are singular values of $\mathbf{Z}$.

- Gaussian map: with high probability all r-rank matrices can be reconstructed provided $m \geq C_M r \max\{n_1, n_2\}$.

Compressive sensing - tensor case

- $\mathbf{\Phi} : \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d} \to \mathbb{R}^m$, $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$, $\mathbf{b} = \mathbf{\Phi}(\mathbf{X}) \in \mathbb{R}^m$, $m \ll n_1 n_2 \cdots n_d$, $\mathbf{X}$ of low rank
- Goal: Reconstruct $\mathbf{X}$ from $\mathbf{\Phi}$ and $\mathbf{b}$
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- convex optimization problem

$$\min_{\mathbf{Z} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}} \|\mathbf{Z}\| \text{ such that } \mathbf{\Phi}(\mathbf{Z}) = \mathbf{b}.$$

- What is a tensor rank?
- What norm should we use?
- The number of measurements for recovery?

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- $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is a rank one tensor:
  $\exists\, \mathbf{u}_1 \in \mathbb{R}^{n_1}, \mathbf{u}_2 \in \mathbb{R}^{n_2}, \ldots, \mathbf{u}_d \in \mathbb{R}^{n_d}$ s.t.

$$\mathbf{X} = \mathbf{u}_1 \otimes \mathbf{u}_2 \otimes \cdots \otimes \mathbf{u}_d, \text{ i.e., } \mathbf{X}(i_1, i_2, \ldots, i_d) = \mathbf{u}_1(i_1)\, \mathbf{u}_2(i_2) \cdots \mathbf{u}_d(i_d).$$

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The rank of a tensor $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ is the smallest number of rank one tensors that sum up to $\mathbf{X}$.

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- **Nuclear tensor norm**: analog of matrix nuclear/trace norm

$$\|\mathbf{X}\|_* = \inf\left\{ \sum_{k=1}^{r} |c_k| : \mathbf{X} = \sum_{k=1}^{r} c_k \mathbf{u}_1^k \otimes \mathbf{u}_2^k \otimes \cdots \otimes \mathbf{u}_d^k, \; r \in \mathbb{N}, \|\mathbf{u}_i^k\|_{\ell_2} = 1, i \in [d], k \in [r] \right\}.$$

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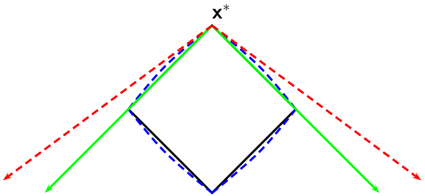
Relaxation



Proposition ([1])

$\hat{\mathbf{x}} = \mathbf{x}^*$ is the unique optimal solution of

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}} \|\mathbf{x}\| \quad s.t.\ \mathbf{b} = \mathbf{\Phi x}$$

if and only if

$$\ker(\mathbf{\Phi}) \cap \mathcal{T}(\mathbf{x}^*) = \{\mathbf{0}\}.$$

- Tangent cone: $\mathcal{T}(\mathbf{x}) = \text{cone}(\mathbf{z} - \mathbf{x} : \|\mathbf{z}\| \leq \|\mathbf{x}\|)$

[1] V. Chandrasekaran, B. Recht, P. A. Parrilo, A. S. Willsky: *The Convex Geometry of Linear Inverse Problems*, 2012.

Convex relaxation via real algebraic geometry

- $\mathbb{R}[\mathbf{x}]$: set of all polynomials in variables $x_1, x_2, \ldots, x_n$
- I ideal in $\mathbb{R}[\mathbf{x}]$
- $\mathbb{R}[\mathbf{x}]_k$: set of polynomials in $\mathbb{R}[\mathbf{x}]$ of degree at most k
- real variety: $\nu_{\mathbb{R}}(I) = \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) = 0, \text{ for all } f \in I\}$

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- Linear programming: $\mathcal{S} = \{\mathbf{x} : \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$

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1. **Lassere's method**

$$\deg\left(h_i\right) \leq k, \quad g(\mathbf{x}) = \sum_{i=1}^{m} g_j\left(\mathbf{x}\right) f_j\left(\mathbf{x}\right),$$

$$I = \langle f_1, \ldots, f_m \rangle, \; g_j f_j \in \mathbb{R}\left[\mathbf{x}\right]_{2k}$$

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Let $I \subseteq \mathbb{R}[\mathbf{x}]$ be an ideal. For a positive integer k, the k-th theta body of an ideal I is

$$\mathrm{TH}_k(I) := \{\mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \geq 0 \text{ for every } f \text{ affine and } k\text{-sos mod } I\}.$$

- $\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \cdots \supseteq \overline{\mathrm{conv}(\nu_{\mathbb{R}}(I))}$
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- $B_{\|\cdot\|_{\theta_1}} = \mathrm{TH}_1(I) \Rightarrow \|\mathbf{X}\|_{\theta_1} = \{\inf_t t \text{ s.t. } \mathbf{X} \in t\,\mathrm{TH}_1(I)\}$

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- alternative approach relies on computing a Groebner basis for the ideal and results in semidefinite program for
  - computing θ_1-norm: $\|\mathbf{X}\|_{\theta_1}$, for all $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$
  - low rank tensor recovery:

$$\min_{\mathbf{Z} \in \mathbb{R}^{n_1 \times n_2 \times n_3}} \|\mathbf{Z}\|_{\theta_1} \quad \text{s.t.} \quad \mathbf{\Phi}(\mathbf{Z}) = \mathbf{b}$$

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  $$X_{21}X_{12} - X_{11}X_{22} = 0, \quad X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 = 1$$

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- $I = \left\{ g \cdot (X_{21}X_{12} - X_{11}X_{22}) + h \cdot \left(\sum_{i=1}^2 \sum_{j=1}^2 X_{ij}^2 - 1 \right) : g, h \in \mathbb{R}[\mathbf{x}] \right\}$

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- $I = \left\langle X_{21}X_{12} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \right\rangle$

1. $\mathbf{x}^{\boldsymbol{\alpha}} >_{grevlex} \mathbf{x}^{\boldsymbol{\beta}}$ if $|\boldsymbol{\alpha}| > |\boldsymbol{\beta}|$ or $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$ and the right-most nonzero entry of $\boldsymbol{\alpha} - \boldsymbol{\beta} \in \mathbb{Z}^n_{\geq 0}$ is negative.

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Definition (Groebner basis)

Fix a monomial order. A basis $G = \{g_1, g_2, \ldots, g_s\}$ of a polynomial ideal $I \subset \mathbb{R}[\mathbf{x}]$ is a Groebner basis (or standard basis) if for all $f \in \mathbb{R}[\mathbf{x}]$ there exist **unique** $r \in \mathbb{R}[\mathbf{x}]$ and $g \in I$ s.t.

$$f = g + r$$

and no monomial of r is divisible by any of the leading monomials in G.

Computation of $\mathrm{TH}_k(I)$

Computation of $\mathrm{TH}_k(I)$

1. Define $\nu_{\mathbb{R}}(I)$ and a corresponding ideal I.

Computation of $\mathrm{TH}_k(I)$

1. $I = \langle X_{12}X_{21} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \rangle$

Computation of $\mathrm{TH}_k(I)$

1. $I = \langle X_{12}X_{21} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \rangle$
2. Find a Groebner basis for the ideal I (with respect to some monomial ordering).

Computation of $\mathrm{TH}_k(I)$

1. $I = \left\langle X_{12}X_{21} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \right\rangle$.

Computation of $\mathrm{TH}_k(I)$

1. $I = \langle X_{12}X_{21} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \rangle$.
2. Find a **θ-basis** $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots = \{f_0 + I, f_1 + I, \ldots\}$ of $\mathbb{R}[\mathbf{x}]/I$, where
   - $\mathcal{B}_1 = \{1 + I, x_1 + I, \ldots, x_n + I\}$

Computation of $\mathrm{TH}_k(I)$

1. $I = \langle X_{12}X_{21} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \rangle$.
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   - $\mathcal{B}_1 = \{1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I\}$

Computation of $\mathrm{TH}_k(I)$

1. $I = \langle X_{12}X_{21} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \rangle$.
2. Find a θ-**basis** $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots = \{f_0 + I, f_1 + I, \ldots\}$ of $\mathbb{R}[\mathbf{x}]/I$, where
   - $\mathcal{B}_1 = \{1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I\}$
   - if $\deg(f_i + I), \deg(f_j + I) \leq k \Rightarrow f_i f_j + I$ is in the $\mathbb{R}$-span of $\mathcal{B}_{2k}$

Computation of $\mathrm{TH}_k(I)$

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   - $\mathcal{B}_1 = \{1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I\}$
   - if $\deg(f_i + I), \deg(f_j + I) \leq k \Rightarrow f_i f_j + I$ is in the $\mathbb{R}$-span of $\mathcal{B}_{2k}$
   - $X_{11}^2 + I = -\left(X_{12}^2 + I\right) - \left(X_{21}^2 + I\right) - \left(X_{22}^2 + I\right) + (1 + I)$

Computation of $\mathrm{TH}_k(I)$

1. $I = \langle X_{12}X_{21} - X_{11}X_{22}, X_{11}^2 + X_{12}^2 + X_{21}^2 + X_{22}^2 - 1 \rangle$.
2. Find a **θ-basis** $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \ldots = \{f_0 + I, f_1 + I, \ldots\}$ of $\mathbb{R}[\mathbf{x}]/I$, where
   - $\mathcal{B}_1 = \{1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I\}$
   - $$\begin{aligned}\mathcal{B}_2 = \mathcal{B}_1 \cup \{&X_{12}^2 + I, X_{21}^2 + I, X_{22}^2 + I, X_{11}X_{12} + I,\\ &X_{11}X_{21} + I, X_{11}X_{22} + I, X_{12}X_{22} + I, X_{21}X_{22} + I\}\end{aligned}$$

Computation of $\mathrm{TH}_k\left(I\right)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_k}\left(\mathbf{X}, \mathbf{y}\right)$.

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_k}(\mathbf{X}, \mathbf{y})$.
   - $[\mathbf{x}]_{\mathcal{B}_k}^T = \{$all elements of $\mathcal{B}_k$ in order$\}$

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_1}(\mathbf{X}, \mathbf{y})$.
   - $[\mathbf{x}]_{\mathcal{B}_1}^T = [1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I]$

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_1}(\mathbf{X}, \mathbf{y})$.
   - $[\mathbf{x}]_{\mathcal{B}_1}^T = [1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I]$
   - $\mathbf{X}_{\mathcal{B}_k} = [\mathbf{x}_{\mathcal{B}_k}][\mathbf{x}_{\mathcal{B}_k}]^T \Rightarrow [\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_1}(\mathbf{X}, \mathbf{y})$.
   - $[\mathbf{x}]_{\mathcal{B}_1}^T = [1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I]$
   - $\mathbf{X}_{\mathcal{B}_k} = [\mathbf{x}_{\mathcal{B}_k}][\mathbf{x}_{\mathcal{B}_k}]^T \Rightarrow [\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1+I & X_{11}+I & X_{12}+I & X_{21}+I & X_{22}+I \\ & X_{11}^2+I & X_{11}X_{12}+I & X_{11}X_{21}+I & X_{11}X_{22}+I \\ & & X_{12}^2+I & X_{12}X_{21}+I & X_{12}X_{22}+I \\ & & & X_{21}^2+I & X_{21}X_{22}+I \\ & & & & X_{22}^2+I \end{bmatrix}$$

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_1}(\mathbf{X}, \mathbf{y})$.
   - $[\mathbf{x}]_{\mathcal{B}_1}^T = [1+I, X_{11}+I, X_{12}+I, X_{21}+I, X_{22}+I]$
   - $\mathbf{X}_{\mathcal{B}_k} = [\mathbf{x}_{\mathcal{B}_k}][\mathbf{x}_{\mathcal{B}_k}]^T \Rightarrow [\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & X_{11}^2 & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{12}X_{21} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_1}(\mathbf{X}, \mathbf{y})$.
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   - $\mathbf{X}_{\mathcal{B}_k} = [\mathbf{x}_{\mathcal{B}_k}][\mathbf{x}_{\mathcal{B}_k}]^T \Rightarrow [\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l+I\in\mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & \boxed{X_{11}^2} & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & \boxed{X_{12}X_{21}} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_1}(\mathbf{X}, \mathbf{y})$.
   - $[\mathbf{x}]_{\mathcal{B}_1}^T = [1 + I, X_{11} + I, X_{12} + I, X_{21} + I, X_{22} + I]$
   - $\mathbf{X}_{\mathcal{B}_k} = [\mathbf{x}_{\mathcal{B}_k}][\mathbf{x}_{\mathcal{B}_k}]^T \Rightarrow [\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$

$$X_{11}^2 + I = -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 + I$$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & \boxed{X_{11}^2} & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & \boxed{X_{12}X_{21}} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

Computation of $\mathrm{TH}_k(I)$

1. Compute a combinatorial moment matrix $M_{\mathcal{B}_1}(\mathbf{X},\mathbf{y})$.
   - $[\mathbf{x}]_{\mathcal{B}_1}^T = [1+I, X_{11}+I, X_{12}+I, X_{21}+I, X_{22}+I]$
   - $\mathbf{X}_{\mathcal{B}_k} = [\mathbf{x}_{\mathcal{B}_k}][\mathbf{x}_{\mathcal{B}_k}]^T \Rightarrow [\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l+I\in\mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$

$$X_{11}^2 + I = -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 + I$$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & \boxed{X_{11}^2} & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & \boxed{X_{12}X_{21}} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

$$X_{12}X_{21} + I = X_{11}X_{22} + I$$

Computation of $\mathrm{TH}_k(I)$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$

Computation of $\mathrm{TH}_k(I)$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $[\mathbf{M}_{\mathcal{B}_k}(\mathbf{y})]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$

Computation of $\mathrm{TH}_k(I)$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} \boxed{1} & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + \boxed{1} & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $[\mathbf{M}_{\mathcal{B}_k}(\mathbf{y})]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$

Computation of $\mathrm{TH}_k(I)$

y_0

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} \boxed{1} & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + \boxed{1} & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $[\mathbf{M}_{\mathcal{B}_k}(\mathbf{y})]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$

Computation of $\mathrm{TH}_k\,(I)$

y_0

$$
\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix}
\boxed{1} & \boxed{X_{11}} & \boxed{X_{12}} & \boxed{X_{21}} & \boxed{X_{22}} \\
 & -X_{12}^2 - X_{21}^2 - X_{22}^2 + \boxed{1} & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\
 & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\
 & & & X_{21}^2 & X_{21}X_{22} \\
 & & & & X_{22}^2
\end{bmatrix}
$$

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $[\mathbf{M}_{\mathcal{B}_k}(\mathbf{y})]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$

Computation of $\mathrm{TH}_k(I)$

y_0 $\qquad\qquad\qquad$ y_1

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} \boxed{1} & \boxed{X_{11}} & \boxed{X_{12}} & \boxed{X_{21}} & \boxed{X_{22}} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + \boxed{1} & \boxed{X_{11}X_{12}} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $[\mathbf{M}_{\mathcal{B}_k}(\mathbf{y})]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$

Computation of $\mathrm{TH}_k\,(I)$

$$y_0 \qquad\qquad y_1 \qquad y_2$$

$$\mathbf{X}_{\mathcal{B}_1} = \left[\begin{array}{ccccc} \boxed{1} & \boxed{X_{11}} & \boxed{X_{12}} & \boxed{X_{21}} & \boxed{X_{22}} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + \boxed{1} & \boxed{X_{11}X_{12}} & \boxed{X_{11}X_{21}} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{array}\right]$$

1. $\left[\mathbf{X}_{\mathcal{B}_k}\right]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^{l} (f_l + I)$
2. $\left[\mathbf{M}_{\mathcal{B}_k}\,(\mathbf{y})\right]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^{l} y_l$

Computation of $\mathrm{TH}_k\,(I)$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

$y_0 \quad y_1 \quad y_2 \quad y_3$

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $[\mathbf{M}_{\mathcal{B}_k}(\mathbf{y})]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$

Computation of $\mathrm{TH}_k\left(I\right)$

$y_0 \quad y_1 \quad y_2 \quad y_3 \quad y_4$

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

1. $\left[\mathbf{X}_{\mathcal{B}_k}\right]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^{l} \left(f_l + I\right)$
2. $\left[\mathbf{M}_{\mathcal{B}_k}\left(\mathbf{X}, \mathbf{y}\right)\right]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^{l} y_l$
3. $Q_{\mathcal{B}_k}\left(I\right) = \pi_{\mathbf{X}}\left\{\left(\mathbf{X}, \mathbf{y}\right) \in \mathbb{R}^{\mathcal{B}_{2k}} : \mathbf{M}_{\mathcal{B}_k}\left(\mathbf{X}, \mathbf{y}\right) \succeq 0, y_0 = 1\right\}$

Computation of $\mathrm{TH}_k\left(I\right)$

y_0 y_1 y_2

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

y_4 y_3

1. $\left[\mathbf{X}_{\mathcal{B}_k}\right]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $\left[\mathbf{M}_{\mathcal{B}_k}\left(\mathbf{X}, \mathbf{y}\right)\right]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$
3. $Q_{\mathcal{B}_k}\left(I\right) = \pi_{\mathbf{X}}\left\{(\mathbf{X}, \mathbf{y}) \in \mathbb{R}^{\mathcal{B}_{2k}} : \mathbf{M}_{\mathcal{B}_k}\left(\mathbf{X}, \mathbf{y}\right) \succeq 0, y_0 = 1\right\}$

Theorem (Gouveia et al., 2010)

$\mathrm{TH}_k\left(I\right) = \overline{Q_{\mathcal{B}_k}\left(I\right)}$

Computation of $\mathrm{TH}_k(I)$

y_0 y_1 y_2

$$\mathbf{X}_{\mathcal{B}_1} = \begin{bmatrix} 1 & X_{11} & X_{12} & X_{21} & X_{22} \\ & -X_{12}^2 - X_{21}^2 - X_{22}^2 + 1 & X_{11}X_{12} & X_{11}X_{21} & X_{11}X_{22} \\ & & X_{12}^2 & X_{11}X_{22} & X_{12}X_{22} \\ & & & X_{21}^2 & X_{21}X_{22} \\ & & & & X_{22}^2 \end{bmatrix}$$

y_4 y_3

1. $[\mathbf{X}_{\mathcal{B}_k}]_{i,j} = f_i f_j + I = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l (f_l + I)$
2. $[\mathbf{M}_{\mathcal{B}_k}(\mathbf{X}, \mathbf{y})]_{i,j} = \sum_{f_l + I \in \mathcal{B}_{2k}} \lambda_{i,j}^l y_l$
3. $Q_{\mathcal{B}_k}(I) = \pi_{\mathbf{X}} \left\{ (\mathbf{X}, \mathbf{y}) \in \mathbb{R}^{\mathcal{B}_{2k}} : \mathbf{M}_{\mathcal{B}_k}(\mathbf{X}, \mathbf{y}) \succeq 0, y_0 = 1 \right\}$

Theorem (Gouveia et al., 2010)

$\mathrm{TH}_k(I) = \overline{Q_{\mathcal{B}_k}(I)}$

4. $B_{\|\cdot\|_{\theta_1}} = \mathrm{TH}_1(I) \Rightarrow \|\mathbf{X}\|_{\theta_1} = \left\{ \inf_t t \text{ s.t. } \mathbf{X} \in t\,\mathrm{TH}_1(I) \right\}$

Semidefinite program

- Given the moment matrix $\mathbf{M}_{\mathcal{B}_k}(\mathbf{X}, \mathbf{y})$ computing the θ_k-norm of a given matrix $\mathbf{X}$ is equivalent to the semidefinite program

$$\min_{\mathbf{y}} t \quad \text{subject to} \quad \mathbf{M}_{\mathcal{B}_k}(\mathbf{X}, \mathbf{y}) \succcurlyeq 0, y_0 = t.$$

- Recovery problem: the θ_k-norm minimization is equivalent to the semidefinite program

$$\min_{\mathbf{y}, \mathbf{Z}} t \quad \text{subject to } \mathbf{M}_{\mathcal{B}_k}(\mathbf{Z}, \mathbf{y}) \succeq 0, \ y_0 = t \quad \text{and} \quad \mathbf{\Phi}(\mathbf{Z}) = \mathbf{b}.$$

- A semidefinite program for computing the norm of a matrix $\mathbf{X} \in \mathbb{R}^{2\times 2}$:

$$\inf_{t,\mathbf{y}} t \quad \text{s.t.} \quad \underbrace{\begin{bmatrix} t & X_{11} & X_{12} & X_{21} & X_{22} \\ X_{11} & -y_4 - y_6 - y_8 + t & y_1 & y_2 & y_3 \\ X_{12} & y_1 & y_4 & y_3 & y_5 \\ X_{21} & y_2 & y_3 & y_6 & y_7 \\ X_{22} & y_3 & y_5 & y_7 & y_8 \end{bmatrix}}_{\text{Moment matrix}} \succeq 0$$

- $3 \times 3 \times 3$ - minimization over 216 variables (0.3705 s)
- $4 \times 4 \times 4$ - minimization over 1000 variables (7.2818 s)
- $5 \times 5 \times 5$ - minimization over 3375 variables (138.904 s)
- $6 \times 6 \times 6$ - minimization over 9261 variables

Matrix case

- $\nu_{\mathbb{R}}(I_M) = \{\mathbf{X} \in \mathbb{R}^{n_1 \times n_2} : \operatorname{rank}(\mathbf{X}) = 1, \|\mathbf{X}\|_F = 1\}$.

$$I_M = \left\{ \sum_{i<k} \sum_{j<l} g_{ijkl} \cdot (X_{il}X_{kj} - X_{ij}X_{kl}) + h \cdot \left(\sum_{i=1}^{n_1} \sum_{j=1}^{n_2} X_{ij}^2 - 1 \right) : g_{ijkl}, h \in \mathbb{R}[X_{11}, \ldots, X_{n_1 n_2}] \right\}$$

Theorem (Rauhut, S., 2015)

For the ideal I_M, $B_{\|\cdot\|_*} = B_{\|\cdot\|_{\theta_1}}$.

Basis for the ideal I

First suggestion for defining an ideal I [1]

- A rank one tensor $\mathbf{X}$: $\mathbf{X} = \mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$
- $\nu_{\mathbb{R}}(I) = \{(\mathbf{X}, \mathbf{u}, \mathbf{v}, \mathbf{w}) : f(\mathbf{X}, \mathbf{u}, \mathbf{v}, \mathbf{w}) = 0, \text{ for all } f \in I\}$
- $\{$rank one, Frobenius norm one tensors$\}$ $= \{\mathbf{X} : (\mathbf{X}, \mathbf{u}, \mathbf{v}, \mathbf{w}) \in \nu_{\mathbb{R}}(I)\}$

$$B_1 = \Big\{ g^{ijk} = X_{ijk} - u_i v_j w_k, i \in [n_1], j \in [n_2], k \in [n_3], \\ h_1 = \sum_i u_i^2 - 1, h_2 = \sum_j v_j^2 - 1, h_3 = \sum_k w_k^2 - 1, \\ g^{ijk}, h_1, h_2, h_3 \in \mathbb{R}[\mathbf{X}, \mathbf{u}, \mathbf{v}, \mathbf{w}] \Big\}$$

[1] V. Chandrasekaran, B. Recht, P. A. Parrilo, A. S. Willsky: *The Convex Geometry of Linear Inverse Problems*, 2012.

Basis for the ideal II

Our suggestion for defining an ideal I

- $\nu_{\mathbb{R}}(I) = \{\mathbf{X} : f(\mathbf{X}) = 0, \text{ for all } f \in I\}$
- $\{\text{rank one, Frobenius norm one tensors}\} = \{\mathbf{X} : \mathbf{X} \in \nu_{\mathbb{R}}(I)\}$

$$\begin{aligned} B_2 = \Big\{ & f_1^{ijk\hat{i}\hat{j}\hat{k}} = -X_{ijk}X_{\hat{i}\hat{j}\hat{k}} + X_{ij\hat{k}}X_{\hat{i}\hat{j}k}, i < \hat{i}, j \leq \hat{j}, k < \hat{k}, \\ & f_2^{ijk\hat{i}\hat{j}\hat{k}} = -X_{ijk}X_{\hat{i}\hat{j}\hat{k}} + X_{i\hat{j}k}X_{\hat{i}j\hat{k}}, i \leq \hat{i}, j < \hat{j}, k < \hat{k}, \\ & f_3^{ijk\hat{i}\hat{j}\hat{k}} = -X_{ijk}X_{\hat{i}\hat{j}\hat{k}} + X_{i\hat{j}\hat{k}}X_{\hat{i}jk}, i < \hat{i}, j < \hat{j}, k \leq \hat{k}, \\ & g = \sum_{i,j,k} X_{ijk}^2 - 1, \ f_1^{ijk\hat{i}\hat{j}\hat{k}}, f_2^{ijk\hat{i}\hat{j}\hat{k}}, f_3^{ijk\hat{i}\hat{j}\hat{k}}, g \in \mathbb{R}[\mathbf{X}] \Big\} \end{aligned}$$

	$\mathbf{X} \in \mathbb{R}^{2\times2\times2}$	$\|\mathbf{X}^{\{1\}}\|_*$	$\|\mathbf{X}^{\{2\}}\|_*$	$\|\mathbf{X}^{\{3\}}\|_*$	$\|\mathbf{X}\|_{\theta_1}$
1	$\left[\begin{array}{cc\|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$	2	2	2	2
2	$\left[\begin{array}{cc\|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right]$	2	2	$\sqrt{2}$	2
3	$\left[\begin{array}{cc\|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$	2	$\sqrt{2}$	2	2
4	$\left[\begin{array}{cc\|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$	$\sqrt{2}$	2	2	2
5	$\left[\begin{array}{cc\|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right]$	$\sqrt{2}+1$	$\sqrt{2}+1$	$\sqrt{2}+1$	3

Table: Tensors $\mathbf{X} \in \mathbb{R}^{2\times2\times2}$ are represented as $\mathbf{X} = [\mathbf{X}(:,:,1) \,|\, \mathbf{X}(:,:,2)]$.

$$\mathbf{X}(i_1, i_2, i_3) = \mathbf{X}^{\{1\}}(i_1, (i_2, i_3)) = \mathbf{X}^{\{2\}}(i_2, (i_1, i_3)) = \mathbf{X}^{\{3\}}(i_3, (i_1, i_2))$$

Recovery of order-3 tensors

- $\mathbf{X}^* = \arg\min_{\mathbf{Z}:\mathbf{\Phi}(\mathbf{Z})=\mathbf{\Phi}(\mathbf{X})} \|\mathbf{Z}\|_{\theta_1}$
- $\mathbf{\Phi} : \mathbb{R}^{n_1 \times n_2 \times n_3} \to \mathbb{R}^m$ Gaussian, $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2 \times n_3}$
- Tensor $\mathbf{X}$ is recovered if $\left|(\mathbf{X} - \mathbf{X}^*)(i,j,k)\right| < 10^{-6}$, $\forall\ i,j,k$.
- Fixed dim, rank and m: 200 trials
- m_{max} maximal m s.t. recovery: 0/200
- m_{min} minimal m s.t. recovery: 200/200

	rank	$m_{\max}$	$m_{\min}$	deg. of freedom
$2 \times 2 \times 3$	1	4	12	12
$3 \times 3 \times 3$	1	7	21	27
$3 \times 4 \times 5$	1	10	31	60
$4 \times 4 \times 4$	1	12	34	64
$4 \times 5 \times 6$	1	18	42	120
$5 \times 5 \times 5$	1	18	43	125
$3 \times 4 \times 5$	2	38	47	60
$4 \times 4 \times 4$	2	31	51	64
$4 \times 5 \times 6$	2	41	85	120

	$\mathbf{X} \in \mathbb{R}^{2\times2\times2\times2}$	$\lVert\mathbf{X}^{\{1\}}\rVert_*$, $\lVert\mathbf{X}^{\{2\}}\rVert_*$, $\lVert\mathbf{X}^{\{3\}}\rVert_*$, $\lVert\mathbf{X}^{\{4\}}\rVert_*$	$\lVert\mathbf{X}\rVert_{\theta_1}$, $\lVert\mathbf{X}\rVert_{u,\theta_1}$
1	$\mathbf{X}(:,:,:,1) = \left[\begin{array}{cc\|cc} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $\mathbf{X}(:,:,:,2) = \left[\begin{array}{cc\|cc} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$	$1+\sqrt{6}$, $2+\sqrt{3}$, $\sqrt{2}+\sqrt{5}$, $\sqrt{2}+\sqrt{5}$	5 , 4.2361
2	$\mathbf{X}(:,:,:,1) = \left[\begin{array}{cc\|cc} 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $\mathbf{X}(:,:,:,2) = \left[\begin{array}{cc\|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$	$1+\sqrt{6}$, $\sqrt{2}+\sqrt{5}$, $2+\sqrt{3}$, $\sqrt{2}+\sqrt{5}$	5 , 4.2361
3	$\mathbf{X}(:,:,:,1) = \left[\begin{array}{cc\|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$, $\mathbf{X}(:,:,:,2) = \left[\begin{array}{cc\|cc} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right]$	$2+\sqrt{3}$, $\sqrt{2}+\sqrt{5}$, $1+\sqrt{6}$, $\sqrt{2}+\sqrt{5}$	5 , 4.2361
4	$\mathbf{X}(:,:,:,1) = \left[\begin{array}{cc\|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$, $\mathbf{X}(:,:,:,2) = \left[\begin{array}{cc\|cc} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{array}\right]$	$\sqrt{3}+\sqrt{5}$, $\sqrt{2}+\sqrt{6}$, $\sqrt{2}+\sqrt{6}$, $\sqrt{3}+\sqrt{5}$	6 , 4.6503
5	$\mathbf{X}(:,:,:,1) = \left[\begin{array}{cc\|cc} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{array}\right]$, $\mathbf{X}(:,:,:,2) = \left[\begin{array}{cc\|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right]$	$\sqrt{2}+\sqrt{6}$, $\sqrt{3}+\sqrt{5}$, $\sqrt{3}+\sqrt{5}$, $\sqrt{2}+\sqrt{6}$	6 , 4.6503

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Thanks for your attention!