Polytopes of Eigensteps of Finite Equal Norm Tight Frames

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Definition

A *finite frame* for a Hilbert space $\mathcal{H}$ of dimension $\dim \mathcal{H} = d$ is a sequence of vectors $F = (f_i)_{i=1}^N$ in $\mathcal{H}$ for which there exist *frame bounds* $0 < A \leq B < \infty$ such that, for every $x \in \mathcal{H}$,

$$A\|x\|^2 \leq \sum_{i=1}^{N} |\langle x, f_i \rangle|^2 \leq B\|x\|^2.$$

- The frame is *equal norm*, if all $\|f_i\|$ are equal.
- The frame is *tight*, if $A = B$ is possible.

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Remark

For finite vector configurations in finite dimensional Hilbert spaces, being a frame is equivalent to being a spanning set.

- A frame F comes with a *frame operator* S_F:

$$S_F \colon \mathcal{H} \longrightarrow \mathcal{H}, \qquad x \longmapsto \sum_{i=1}^{N} \langle x, f_i \rangle f_i.$$

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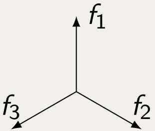
- The frame operator encodes essential properties of the frame. In particular, the smallest and largest eigenvalues of S_F are the optimal frame bounds of F.
- This implies that F is a tight frame if and only if $S_F = \lambda \cdot \mathrm{id}_{\mathcal{H}}$ for some $\lambda \neq 0$.

Example (Mercedes-Benz frame)

Let $\mathcal{H} = \mathbb{R}^2$ and consider the vector configuration

$$F = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix} = \sqrt{\frac{2}{3}} \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}.$$



The frame operator is $FF^* = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In other words, every $x \in \mathbb{R}^2$ reconstructs as

$$x = \sum_{i=1}^{3} \langle x, f_i \rangle f_i,$$

just like for orthonormal bases! (Such F is called a *Parseval frame*)

Problem

Given $(\mu_n)_{n=1}^N$, $(\lambda_i)_{i=1}^d$ *non-increasing sequences of non-negative real numbers, find all matrices* $F = (f_n)_{n=1}^N$ *such that*

- $\|f_n\|^2 = \mu_n$ *for all* n,
- *the spectrum of* FF^* *is* $(\lambda_i)_{i=1}^d$.

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Solved in a 2013 paper by Cahill, Fickus, Mixon, Poteet and Strawn using an algorithm involving *eigensteps*.

Definition (Eigensteps)

Given a $d \times N$ matrix $F = (f_n)_{n=1}^{N}$ over $\mathbb{C}$ or $\mathbb{R}$, define

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Example

Let $F = \begin{pmatrix} f_1 & f_2 & f_3 \end{pmatrix}$ be the Mercedes-Benz frame. We obtain the spectra $(0,0)$, $(\frac{2}{3},0)$, $(1,\frac{1}{3})$ and $(1,1)$.

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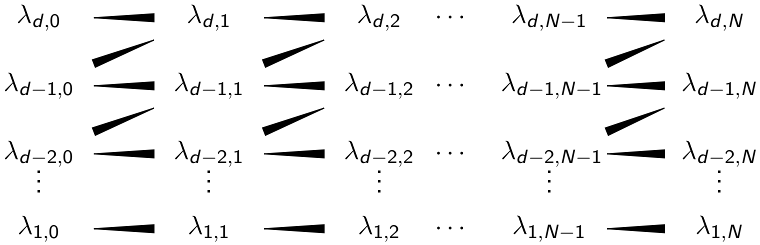
We represent this data in an *eigenstep tableau*

$$\lambda_F = \begin{pmatrix} \lambda_{2,0} & \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} \\ \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \frac{1}{3} & 1 \\ 0 & \frac{2}{3} & 1 & 1 \end{pmatrix}.$$

A theorem by Horn and Johnson states that the spectra of $F_n F_n^*$ and $F_{n+1} F_{n+1}^*$ *interlace*:



A wedge $\lambda_{i,j} \blacktriangleleft \lambda_{k,l}$ denotes an inequality $\lambda_{i,j} \leq \lambda_{k,l}$.

Furthermore, for $0 \le n \le N$ we have the *trace conditions*

$$\sum_{i=1}^{d} \lambda_{i,n} = \mathrm{Tr}(F_n F_n^*) = \mathrm{Tr}(F_n^* F_n) = \sum_{k=1}^{n} \|f_k\|^2 = \sum_{k=1}^{n} \mu_k .$$

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Theorem (Cahill, Fickus, Mixon, Poteet and Strawn 2013)

The following conditions completely characterize the valid sequences of eigensteps for given sequences $(\mu_n)_{n=1}^N$ and $(\lambda_i)_{i=1}^d$:

- *the interlacing conditions,*
- *the trace conditions,*
- $\lambda_{i,0} = 0$ *and* $\lambda_{i,N} = \lambda_i$ *for* $1 \leq i \leq d$.

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- *the interlacing conditions,*
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$\Rightarrow$ The valid sequences of eigensteps form a polytope in $\mathbb{R}^{d\times(N+1)}$.

We only consider equal norm tight frames with norm-squares $\mu = d$ and $FF^* = N \cdot I_d$. In this case the conditions for valid sequences of eigensteps can be summarized as:

$$
\begin{array}{cccccccccccc}
0 = & \lambda_{d,0} & \leq & \lambda_{d,1} & \leq & \lambda_{d,2} & \cdots & \lambda_{d,N-1} & \leq & \lambda_{d,N} & = N \\
 & & \diagup & & \diagup & & & & \diagup & & \\
0 = & \lambda_{d-1,0} & \leq & \lambda_{d-1,1} & \leq & \lambda_{d-1,2} & \cdots & \lambda_{d-1,N-1} & \leq & \lambda_{d-1,N} & = N \\
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\Sigma & 0 & & d & & 2d & \cdots & (N-1)d & & Nd &
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The solutions of this system of linear equations and inequalities form the polytope $\Lambda_{N,d}$ of eigensteps of finite equal norm tight frames.

Questions

- *What is the dimension of* $\Lambda_{N,d}$*?*
- *What are the facet-describing inequalities of* $\Lambda_{N,d}$*?*
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Theorem (Haga, P)

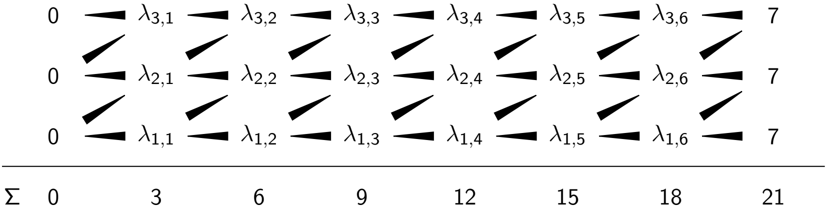
1. *The dimension of* $\Lambda_{N,d}$ *is* 0 *for* $d = 0$ *and* $d = N$*, otherwise*

$$\dim(\Lambda_{N,d}) = (d-1)(N-d-1).$$

2. *For* $2 \leq d \leq N-2$ *the number of facets of* $\Lambda_{N,d}$ *is*

$$d(N-d-1) + (N-d)(d-1) - 2.$$

The defining conditions of $\Lambda_{7,3}$



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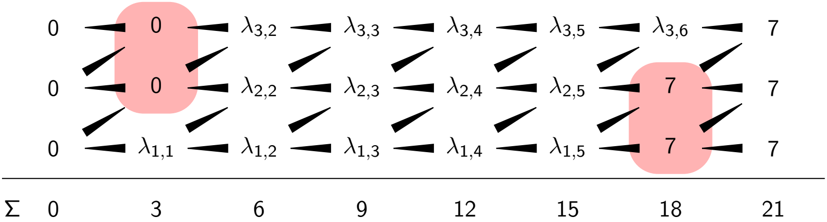

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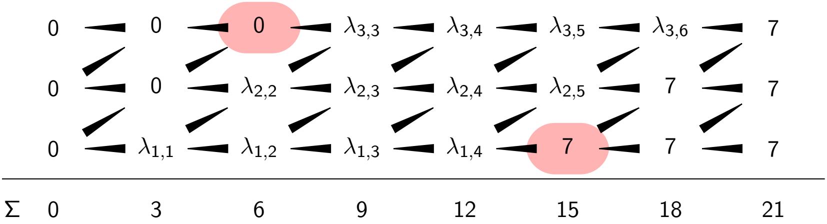

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	0	0	0	$\lambda_{3,3}$	$\lambda_{3,4}$	$\lambda_{3,5}$	$\lambda_{3,6}$	7
	0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\lambda_{2,5}$	7	7
	0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	7	7	7
Σ	0	3	6	9	12	15	18	21

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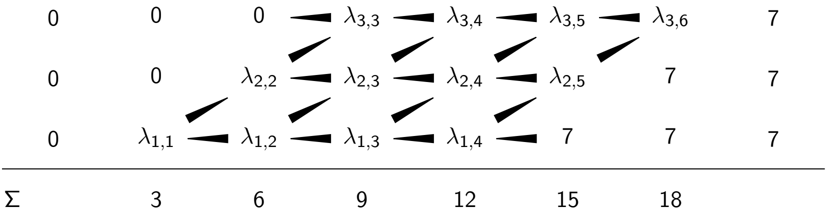

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0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\lambda_{2,5}$	7	7
0	$\lambda_{1,1}$	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	7	7	7
Σ	3	6	9	12	15	18	

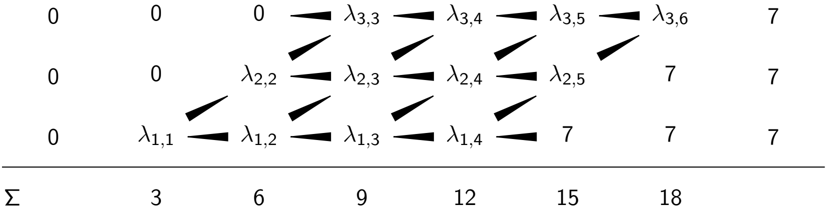
The remaining equations are linearly independent, so

$$\dim(\Lambda_{N,d}) \leq d(N+1) - 2 \cdot \frac{d(d+1)}{2} - (N-1)$$
$$= (d-1)(N-d-1).$$

	0	0	0	1	2	3	4	7
	0	0	2	3	4	5	7	7
	0	3	4	5	6	7	7	7
Σ		3	6	9	12	15	18	

The remaining inequalities can be satisfied strictly by the *special point* $\widehat{\lambda}$, so

$$\dim(\Lambda_{N,d}) = (d-1)(N-d-1).$$



Question

Which of the remaining inequalities define the facets of $\Lambda_{N,d}$?

	0	0	0	$\lambda_{3,3}$	$\lambda_{3,4}$	$\lambda_{3,5}$	4	7
	0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\lambda_{2,5}$	7	7
	0	3	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	7	7	7
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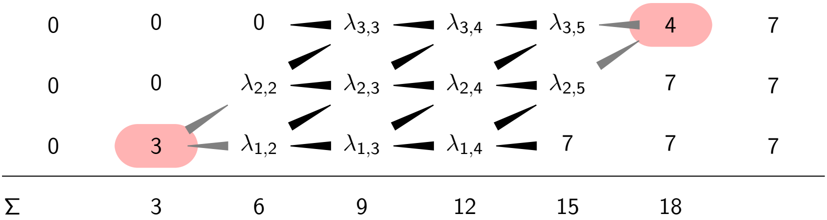
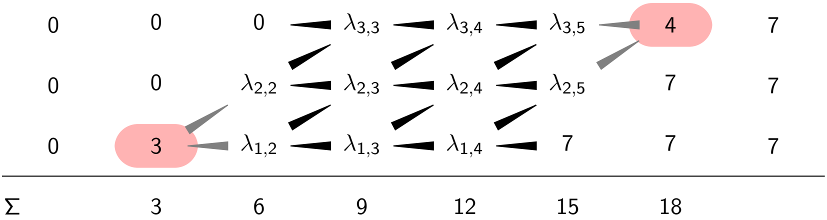


The inequalities $\lambda_{2,2} \leq 3 \leq \lambda_{1,2}$ are implied by $\lambda_{2,2} \leq \lambda_{1,2}$ and $\lambda_{2,2} + \lambda_{1,2} = 6$.



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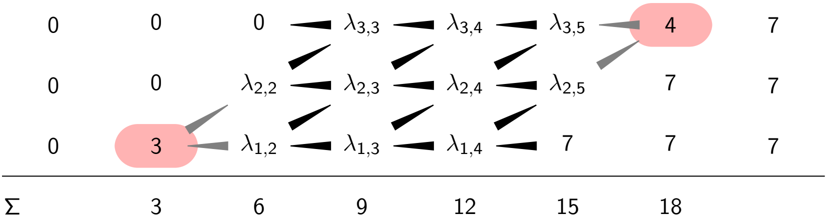

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Similarly, the inequalities $\lambda_{3,5} \leq 4 \leq \lambda_{2,5}$ are redundant.

	0	0	0	$\lambda_{3,3}$	$\lambda_{3,4}$	$\lambda_{3,5}$	4	7
	0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	$\lambda_{2,4}$	$\lambda_{2,5}$	7	7
	0	3	$\lambda_{1,2}$	$\lambda_{1,3}$	$\lambda_{1,4}$	7	7	7
Σ		3	6	9	12	15	18	

Theorem

The remaining

$$d(N-d-1)+(d-1)(N-d)-2$$

inequalities define the facets of $\Lambda_{N,d}$.

In the proof we look at each of the inequalities and construct a point satisfying all conditions except the considered inequality.

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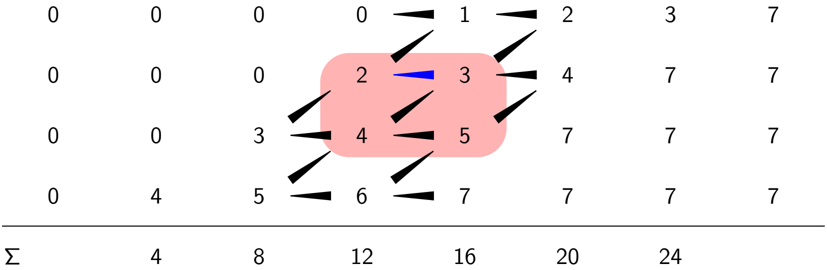
Some examples of those points for $\Lambda_{7,4}$ ...

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

This is the tableau of the special point $\widehat{\lambda} \in \Lambda_{7,4}$.



We want to make only the blue inequality fail by changing the highlighted entries.

	0	0	0	0	1	2	3	7
	0	0	0	3	2	4	7	7
	0	0	3	3	6	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

Only the blue inequality fails, all other conditions are satisfied!

	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
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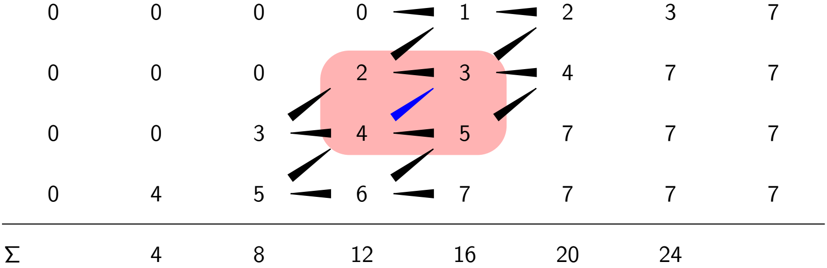

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	0	0	0	0	1	2	3	7
	0	0	0	3	4	4	7	7
	0	0	3	3	4	7	7	7
	0	4	5	6	7	7	7	7
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	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
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	0	0	0	0	1	2	3	7
	0	0	0	3	3	4	7	7
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	0	4	6	6	7	7	7	7
Σ		4	8	12	16	20	24	

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	0	0	0	0	1	2	3	7
	0	0	0	2	3	4	7	7
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This is the tableau of the special point $\widehat{\lambda} \in \Lambda_{7,4}$.

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	0	0	0	2	3	4	7	7
	0	0	3	4	5	7	7	7
	0	4	5	6	7	7	7	7
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We want to make only the blue inequality fail by changing the highlighted entries.

	0	0	0	0	-1	2	3	7
	0	0	0	2	4	4	7	7
	0	0	3	4	6	7	7	7
	0	4	5	6	7	7	7	7
Σ		4	8	12	16	20	24	

Only the blue inequality fails, all other conditions are satisfied!

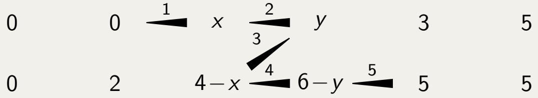
Example ($N = 5, d = 2$)

The polytope $\Lambda_{5,2}$ is 2-dimensional and has 5 facets.

	0	0	$\lambda_{2,2}$	$\lambda_{2,3}$	3	5
	0	2	$\lambda_{1,2}$	$\lambda_{1,3}$	5	5
Σ		2	4	6	8	

Parametrize the polytope by $x = \lambda_{2,2}$, $y = \lambda_{2,3}$ . . .

Example ($N = 5, d = 2$)



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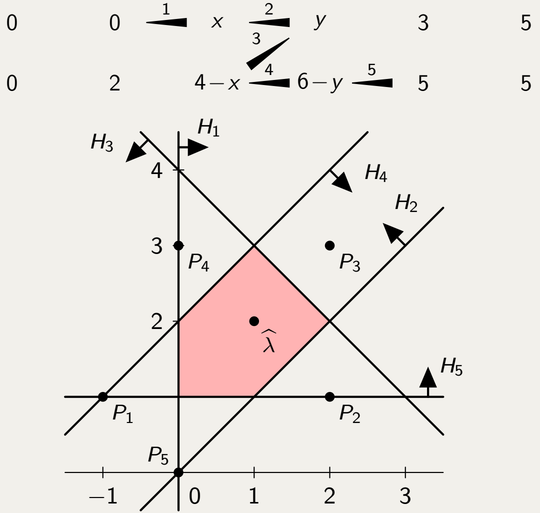

- While studying $\Lambda_{N,d}$ we discovered affine isomorphisms

$$\Phi_{N,d} \colon \Lambda_{N,d} \longrightarrow \Lambda_{N,d},$$
$$\Psi_{N,d} \colon \Lambda_{N,d} \longrightarrow \Lambda_{N,N-d}.$$

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- These isomorphisms are related to reversing the order of frame vectors and taking Naimark complements of frames.

Proposition

There is an affine involution $\Phi_{N,d}\colon \Lambda_{N,d} \longrightarrow \Lambda_{N,d}$ *given by*

$$(\Phi_{N,d}(\lambda))_{i,n} = N - \lambda_{d-i+1,N-n}.$$

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$$\left(\Phi_{N,d}(\lambda)\right)_{i,n} = N - \lambda_{d-i+1,N-n}.$$

Example

For $N = 5$, $d = 3$ the involution $\Phi_{5,3} \colon \Lambda_{5,3} \to \Lambda_{5,3}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & 5-\lambda_{1,2} & 5-\lambda_{1,1} & 5 \\ 0 & 0 & 5-\lambda_{2,3} & 5-\lambda_{2,2} & 5 & 5 \\ 0 & 5-\lambda_{3,4} & 5-\lambda_{3,3} & 5 & 5 & 5 \end{pmatrix}.$$

Proposition

There is an affine isomorphism $\Psi_{N,d}\colon \Lambda_{N,d} \longrightarrow \Lambda_{N,N-d}$ *given by*

$$(\Psi_{N,d}(\lambda))_{i,n} = \begin{cases} \lambda_{d+i-n,N-n}, & \text{for } i \leq n \leq d+i-1, \\ 0, & \text{for } n < i, \\ N, & \text{for } n > d+i-1. \end{cases}$$

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Example

For $N = 5$, $d = 3$ the isomorphism $\Psi_{5,3}\colon \Lambda_{5,3} \to \Lambda_{5,2}$ is given by

$$\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & \lambda_{3,3} & \lambda_{2,2} & \lambda_{1,1} & 5 \\ 0 & \lambda_{3,4} & \lambda_{2,3} & \lambda_{1,2} & 5 & 5 \end{pmatrix}.$$

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Theorem (Haga, P)

The affine isomorphisms $\Phi_{N,d}$ and $\Psi_{N,d}$ satisfy

$$\Phi_{N,d}(\lambda_F) = \lambda_{\widetilde{F}},$$
$$\Psi_{N,d}(\lambda_F) = \lambda_{\widetilde{G}}.$$

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- Are the frame classes belonging to certain classes of eigensteps interesting? ($\widehat{\lambda}$, $\partial\Lambda_{N,d}$, vertices of $\Lambda_{N,d}$, ...?)
- Can we obtain similar non-redundant descriptions of more general polytopes of eigensteps $\Lambda((\mu_n)_{n=1}^N, (\lambda_i)_{i=1}^d)$?

Thanks for your attention!

...and feel free to look into our preprint at `arXiv:1507.04197`