Topological Complexity and related invariants

Lucile Vandembroucq
Centro de Matemática - Universidade do Minho - Portugal
Joint work with J. Calcines and J. Carrasquel

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Topological Complexity

X - configuration space of a mechanical system.

A motion planning algorithm is a section $s : X \times X \to X^I$ ($I = [0, 1]$) of

$$\pi = ev_{0,1} : X^I \to X \times X, \quad \gamma \mapsto (\gamma(0), \gamma(1))$$

$\mathrm{TC}(X) =$"minimal number of rules in a motion planner in X".

From now on X is a path-connected CW-complex.

Definition. (M. Farber, 2003) $TC(X)$ is the least integer n such that $X \times X$ can be covered by n open sets U_1,..., U_n on each of which the fibration

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admits a **continuous** (local) section $s_i : U_i \to X^I$.

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Example. (M. Farber) $\mathrm{TC}(S^n) = \begin{cases} 2 & n \text{ odd} \\ 3 & n \text{ even} \end{cases}$

Theorem. (M. Farber)

$$\left.\begin{array}{c} \mathrm{cat}(X) \\ \mathrm{z.d.cuplength}(X) + 1 \end{array}\right\} \leq \mathrm{TC}(X) \leq \begin{cases} 2\mathrm{cat}(X) - 1 \\ \dim(X) + 1 & (X \text{ 1-conn.}) \end{cases}$$

where

- (Lusternik-Schnirelmann category)

$$\mathrm{cat} X \leq n :\Leftrightarrow X = V_1 \cup \ldots \cup V_n,\ V_i \text{ contractile in } X.$$

- (zero-divisors cuplength)

$$\mathrm{z.d.cuplength}(X) = \mathrm{nil}(\ker \cup)$$

where $\cup : H^*(X) \otimes H^*(X) \to H^*(X)$ is the cup product.

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Monoidal Topological Complexity

Variations of TC have been introduced, for instance:

- Symmetric Topological Complexity (M. Farber, M. Grant, 2006)
- Higher Topological Complexity (Y. Rudyak, 2009)

and also:

Definition. (Monoidal TC - N. Iwase, M. Sakai, 2010)
$TC^M(X)$ is the least integer n such that $X \times X$ can be covered by n open sets $U_1, \ldots, U_n$ on each of which $\pi : X^I \to X \times X$ admits a (continuous) section $s_i : U_i \to X^I$ such that

$$s_i(x,x) = c_x \quad \text{if } (x,x) \in U_i.$$

Theorem. (I-S) $\mathrm{TC}(X) \leq \mathrm{TC}^M(X) \leq \mathrm{TC}(X) + 1$.

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Theorem. (A. Dranishnikov, 2012) I-S conjecture holds when

- $\dim(X) \leq \mathrm{TC}(X)(conn(X)+1)-2$.
- X is a Lie group.

Remark. If I-S conjecture holds, then for any space X,

$$\mathrm{TC}(X) \geq \mathrm{cat}(C_\Delta)$$

where $C_\Delta = X \times X/\Delta(X)$ is the cofibre of $\Delta : X \to X \times X$.

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TC, Sectional Category

Definition. (A. Schwarz, 1966) $\mathrm{secat}(p : E \to B)$ is the least integer n such that B can be covered by n open sets on each of which p admits a (continuous) local section.

- $\mathrm{TC}(X) = \mathrm{secat}(\pi : X^I \to X \times X)$
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  where $P_0X = \{\gamma \in X^I, \gamma(0) = *\}$.

- By requiring *homotopy* sections secat can be defined for any map and we have

$$\mathrm{TC}(X) = \mathrm{secat}(\Delta : X \to X \times X) \qquad \mathrm{cat}(X) = \mathrm{secat}(* \to X)$$

$$\begin{tikzcd} X \arrow[rr, "c_X", "\sim"'] \arrow[dr, "\Delta"'] & & X^I \arrow[dl, "\pi"] \\ & X \times X & \end{tikzcd} \qquad \begin{tikzcd} * \arrow[rr, "\sim"] \arrow[dr] & & P_0X \arrow[dl, "ev_1"] \\ & X & \end{tikzcd}$$

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Sectional category and Joins

The join of 2 fibrations $p : E \to B$ and $p' : E' \to B$ is the map

$$\begin{array}{rcl} E *_B E' := E \amalg (E \times_B E' \times [0,1]) \amalg E' / \sim & \to & B \\ \langle e, e', t \rangle & \mapsto & p(e) = p'(e') \end{array}$$

where $\sim$ is given by $(e, e', t) \sim \begin{cases} e & t = 0 \\ e' & t = 1 \end{cases}$

This map is a fibration with fibre

$$F * F' = F \amalg F \times F' \times [0,1] \amalg F' / \sim$$

where F and F' are the respective fibres of p and p'.

For $p : E \to B$, consider

$$p_1 = p \quad \text{and, for } n \geq 2, \quad p_n : J_n(p) = \underbrace{E *_B \cdots *_B E}_{n \,\text{factors}} \to B$$

Theorem. (A. Schwarz) If B is normal, then

$$\mathrm{secat}(p) \leq n \iff p_n \text{ admits a (continuous) section.}$$

For $p = \pi : X^I \to X \times X$:

Corollary. $\mathrm{TC}(X) \leq n \iff \pi_n : J_n(\pi) \to X \times X$ has a section.

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Given a fibration $p : E \to B$, we have, for any n, a canonical diagram:

$$\begin{tikzcd} E \arrow[r, "\lambda_n"] \arrow[dr, "p"'] & J_n(p) \arrow[d, "p_n"] \\ & B \end{tikzcd}$$

If $p = \pi : X^I \to X \times X$ we have $\begin{tikzcd} X \arrow[r, "c_x"] \arrow[drr, "\Delta"'] & X^I \arrow[r, "\lambda_n"] \arrow[dr, "\pi"] & J_n(\pi) \arrow[d, "\pi_n"] \\ & & X \times X \end{tikzcd}$

Theorem. (Dranishnikov) $\mathrm{TC}^M(X) \leq n$ iff

$$\pi_n : J_n(\pi) \to X \times X \text{ admits a section } s \text{ such that } s\Delta = \lambda_n c_x.$$

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Doerane-El Haouari relative category and conjecture

Definition. (D-EH, 2012) The relative category of a fibration $p : E \to B$ is given by $\mathrm{relcat}(p) \leq n \; :\Longleftrightarrow \; p_n$ admits a section s such that $sp \simeq \lambda_n$.

$$\begin{tikzcd} E \arrow[r, "\lambda_n"] \arrow[dr, "p"'] & J_n(p) \arrow[d, shift left, "p_n"] \\ & B \arrow[u, bend left, "s"] \end{tikzcd}$$

Theorem. (D-EH) $\mathrm{secat}(p) \leq \mathrm{relcat}(p) \leq \mathrm{secat}(p) + 1$.

Conjecture. (D-EH) If p admits a homotopy retraction then

$$\mathrm{relcat}(p) = \mathrm{secat}(p).$$

If $p = ev_1 : P_0X \to X$ we have

$$\begin{tikzcd} * \arrow[r, "\sim"] \arrow[drr] & P_0X \arrow[r, "\lambda_n"] \arrow[dr, "ev_1"] & J_n(ev_1) \arrow[d, "(ev_1)_n"] \\ & & X \end{tikzcd}$$

- ev_1 has a homotopy retraction ($X \to * \xrightarrow{\sim} P_0X$)
- D-EH conjecture holds.

For $p = \pi : X^I \to X \times X$

- there is a homotopy retraction, for instance

$$X \times X \stackrel{pr_1}{\to} X \stackrel{c_X}{\to} X^I$$

- we can prove that

$$\mathrm{relcat}(\pi) = \mathrm{TC}^M(X)$$

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Consequence. For $p = \pi : X^I \to X \times X$

D-EH conjecture = I-S conjecture

Theorem. D-EH conjecture holds after suspension.

Meaning: Suppose that

- p admits a homotopy retraction r
- $\Sigma p_n : \Sigma J_n(p) \to \Sigma(B)$ has a homotopy section s

then

$\Sigma p_n : \Sigma J_n(p) \to \Sigma(B)$ admits a homotopy section $\tilde{s}$ such that $\tilde{s}\Sigma p \simeq \Sigma\lambda_n$

$$\begin{tikzcd} \Sigma E \arrow[r, "\Sigma\lambda_n"] \arrow[dr, "\Sigma p"'] & \Sigma J_n(p) \arrow[d, "\Sigma p_n"] \\ & \Sigma B \end{tikzcd}$$

Corollary. I-S conjecture holds after suspension.

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Corollary. I-S conjecture holds after suspension.

Proof:

- Since $p : E \to B$ admits a homotopy retraction r, the sequence

$$E \xrightarrow{p} B \xrightarrow{q} C_p$$

splits after suspension:

$$\begin{tikzcd} \Sigma E \arrow[r, shift left, "\Sigma p"] & \Sigma B \arrow[l, shift left, "\Sigma r"] \arrow[r, shift left, "\Sigma q"] & \Sigma C_p \arrow[l, shift left, "\nu"] \end{tikzcd} \qquad \nu\Sigma q + \Sigma p \Sigma r \simeq id$$

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- If s is a homotopy section of Σp_n then

$$\tilde{s} := s\nu\Sigma q + \Sigma\lambda_n\Sigma r \qquad \begin{tikzcd} \Sigma E \arrow[r, "\Sigma\lambda_n"] \arrow[dr, "\Sigma p"] & \Sigma J_n(p) \arrow[d, shift right] & \\ & \Sigma B \arrow[ul, bend left, "\Sigma r"] \arrow[u, bend right, "s"'] \arrow[r, "\Sigma q"] & \Sigma C_p \arrow[l, bend left, "\nu"] \end{tikzcd}$$

is a homotopy section of Σp_n such that $\tilde{s}\Sigma p \simeq \Sigma\lambda_n$

Another weak version of I-S conjecture

Considering "weak" versions of cat and TC in the sense of Berstein-Hilton:

Theorem. $\mathrm{wTC}(X) = \mathrm{wTC}^M(X) = \mathrm{wcat}(C_\Delta)$
where:

- $\mathrm{wcat}(C_\Delta) \leq n :\Leftrightarrow C_\Delta \stackrel{\Delta_n}{\to} (C_\Delta)^n \to (C_\Delta)^{\wedge n}$ is homotopically trivial.
- $\mathrm{wTC}(X) \leq n :\Leftrightarrow X \times X \stackrel{\Delta_n}{\to} (X \times X)^n \to (C_\Delta)^n \to (C_\Delta)^{\wedge n}$ is homotopically trivial.
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Rational Homotopy Theory

- Sullivan (contravariant) functor of polynomial forms:
  $A_{PL} : TOP \to CDGA$ (comm. diff. grad. algebra)
  - If X is simply-connected and of finite type then $A_{PL}(X)$ contains all rational homotopy information about X.
  - In particular, $H(A_{PL}(X)) = H^*(X; \mathbb{Q})$.
- Model of X in $CDGA$: (A, d) weakly equivalent to $A_{PL}(X)$:

$$(A, d) \xrightarrow{\sim} \bullet \xleftarrow{\sim} \quad \xrightarrow{\sim} \cdots \xleftarrow{\sim} A_{PL}(X)$$

- Sullivan model of X: $\qquad (\Lambda V, d) \xrightarrow{\sim} A_{PL}(X)$

  If $d(V) \subset \Lambda^{>1}(V)$ the model is said to be *minimal*. In this case $V \cong$ dual of $\pi_*(X) \otimes \mathbb{Q}$.

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Rational Homotopy Theory

- Sullivan (contravariant) functor of polynomial forms:

  $A_{PL} : TOP \to CDGA$ (comm. diff. grad. algebra)

  - If X is simply-connected and of finite type then $A_{PL}(X)$ contains all rational homotopy information about X.
  - In particular, $H(A_{PL}(X)) = H^*(X; \mathbb{Q})$.

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Let $E \xrightarrow{p} B$ be a fibration with E, B simply-connected spaces of finite type.

By applying A_{PL} we get
$$\begin{tikzcd} A_{PL}(E) & A_{PL}(J_n(p)) \arrow[l, "A_{PL}(\lambda_n)"'] \\ & A_{PL}(B) \arrow[u, "A_{PL}(p_n)"'] \arrow[ul, "A_{PL}(p)"] \end{tikzcd}$$

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- secat$_0(p) \leq n$ if $A_{PL}(p_n)$ admits a homotopy retraction in *CDGA*.
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For $p = \pi : X^I \to X \times X$ we use the notation $\mathrm{TC}_0(X)$, $\mathrm{TC}_0^M(X)$.

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Theorem. D-EH conjecture holds at the level of $A_{PL}(E)$-modules.

Theorem. (J. Carrasquel, 2012) Let $\varphi : (A, d) \to (C, d)$ be a surjective model of p. If the projection

$$(A, d) \to (A/(\ker \varphi)^n, \bar{d})$$

admits a homotopy retraction in *CDGA* then $\mathrm{secat}_0(p) \le n$.

- For $p = \pi : X^I \to X \times X$: consider the multiplication

  $$\mu : \Lambda V \otimes \Lambda V \to \Lambda V \qquad (\Lambda V, d) \text{ Sullivan model of } X$$

  If $\Lambda V \otimes \Lambda V \to \Lambda V \otimes \Lambda V/(\ker \mu)^n$ admits a htpy retraction then $\mathrm{TC}_0(X) \le n$. (B. Jessup, P.-E. Parent, A. Murillo, 2012)

- (Y. Félix, S. Halperin, 1982) For $p = ev_1 : P_0X \to X$:

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Corollary. Let φ be a surjective model of p. We have

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In particular, If (A, d) is a model of X with multiplication $\mu_A : A \otimes A \to A$ then

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we can state that I-S conjecture holds rationally for:

- formal spaces: $(H^*(X), 0)$ is a model

$$\mathrm{nil}\,\ker\cup + 1 \leq \mathrm{TC}_0 \leq \mathrm{TC}_0^M \leq \mathrm{nil}\,\ker\cup + 1$$

- spaces whose rational homotopy is concentrated in odd degrees

$$\mathrm{nil}\,\ker\cup + 1 = \mathrm{TC}_0 = \mathrm{TC}_0^M = \mathrm{nil}\,\ker\mu_{\Lambda V} + 1$$

- for the (non formal) space $X = S^3_a \vee S^3_b \cup_{[a,[a,b]]} e^8 \cup_{[b,[a,b]]} e^8$.

$$\mathrm{nil}\,\ker\cup + 1 = 3 \qquad \mathrm{MTC} = \mathrm{TC}_0 = 4 = \mathrm{nil}\,\ker\mu_A + 1$$

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Remarks

- (N. Dupont, 1999) There exists a CW-complex X such that

$$\mathrm{cat}_0(X) < \mathrm{nil}\,\ker \varepsilon_A + 1$$

where $\varepsilon_A : A \to \mathbb{Q}$ is the augmentation of any model (A, d) of X.

- (O. Cornea, Y. Félix, S. Halperin, 1998) If X is a Poincaré duality complex then there exists a model (A, d) of X such that

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