Derived categories arising from combinatorial data

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Perspective

Triangulated and *derived categories* can relate objects of different nature:

- *Coherent sheaves* over algebraic varieties and *modules* over non-commutative algebras [Beilinson 1978, Kapranov 1988]
- *Homological mirror symmetry conjecture* [Kontsevich 1994]

. . . but also relate non-isomorphic objects of the same nature:

- Morita theory for derived categories [Rickard 1989]
- Derived categories of coherent sheaves [Bondal-Orlov 2002]
- *Broué's conjecture* on blocks of group algebras [Broué 1990]

Focus

Derived categories arising from combinatorial objects.

- Partially ordered sets (*posets*)

  $\rightsquigarrow$ diagrams, sheaves, modules over *incidence algebra*

- Quivers with *zero-* and *commutativity-*relations

- Quivers with *potential*

  e.g. Jacobian algebras arising from *surface triangulations*

Path algebras of quivers

A *quiver* Q is a (finite) oriented graph.

Let K be a field. The *path algebra* KQ is the K-algebra

- spanned by all paths in Q,
- with multiplication given by composition of paths.

Example.

$$Q = \bullet_1 \xrightarrow{\alpha} \bullet_2 \xrightarrow{\beta} \bullet_3 \qquad KQ = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$$

$$e_1, e_2, e_3, \alpha, \beta, \alpha\beta \qquad \alpha \cdot \beta = \alpha\beta \quad \beta \cdot \alpha = 0$$

Quivers with relations

A *relation* is a linear combination of paths having the same endpoints.

Examples of relations:

- *zero* relation p

$$\begin{tikzcd} \bullet \arrow[r, "\alpha"] \arrow[rr, bend right, dotted, no head] & \bullet \arrow[r, "\beta"] & \bullet \end{tikzcd}$$

$$\alpha\beta$$

- *commutativity* relation $p - q$

$$\begin{tikzcd} & \bullet \arrow[dr, "\beta"] & \\ \bullet \arrow[ur, "\alpha"] \arrow[dr, "\gamma"'] \arrow[rr, dotted, no head] & & \bullet \\ & \bullet \arrow[ur, "\delta"'] & \end{tikzcd}$$

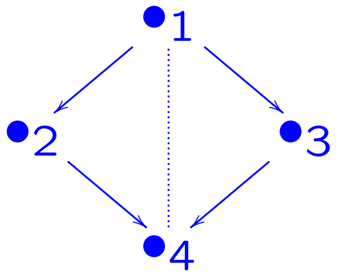
$$\alpha\beta - \gamma\delta$$

- cyclic derivative of a *potential* (linear combination of cycles)

A *quiver* Q *with relations* defines an algebra KQ/I by considering the path algebra KQ modulo the ideal I generated by all the relations.

Example – Posets, diagrams and sheaves

Let $X = \{1, 2, 3, 4\}$ with $1 < 2$, $1 < 3$, $1 < 4$, $2 < 4$, $3 < 4$.



$$KX = \begin{pmatrix} * & * & * & * \\ 0 & * & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix}$$

$$\begin{tikzcd} & V_1 \arrow[dl, "r_{12}"'] \arrow[dr, "r_{13}"] & \\ V_2 \arrow[dr, "r_{24}"'] & & V_3 \arrow[dl, "r_{34}"] \\ & V_4 & \end{tikzcd}$$

$r_{24}r_{12} = r_{34}r_{13}$

X has a natural topology where the *open sets* are

$$\phi, \{4\}, \{2, 4\}, \{3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}.$$

Derived categories

R – ring, $\mathrm{Mod}\, R$ – the category of (right) R-modules.

The *derived category* $\mathcal{D}(\mathrm{Mod}\, R)$ is obtained from the category of complexes of R-modules by formally inverting all the quasi-isomorphisms. It is a *triangulated category*.

A *quasi-isomorphism* is a morphism of complexes $f : K \to L$ inducing isomorphisms $H^i f : H^i K \xrightarrow{\sim} H^i L$ on the cohomology for all $i \in \mathbb{Z}$.

Two rings R, S are *Morita equivalent* if $\mathrm{Mod}\, R \simeq \mathrm{Mod}\, S$. They are *derived equivalent* if $\mathcal{D}(\mathrm{Mod}\, R) \simeq \mathcal{D}(\mathrm{Mod}\, S)$.

How to assess derived equivalence?

No known *algorithm* that decides, given two combinatorial objects X and Y, whether the associated derived categories $\mathcal{D}_X$ and $\mathcal{D}_Y$ are equivalent. *However*, one can consider:

• *Invariants* of derived equivalence;
If $\mathcal{D}_X \simeq \mathcal{D}_Y$, then X and Y must have the same invariants.

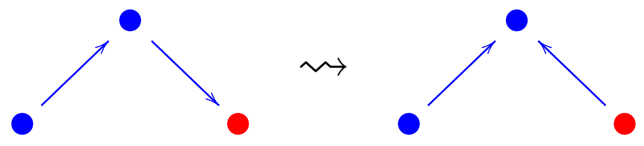
- The *number of points*.
- The *Euler bilinear form*, which for posets is closely related to the *Möbius function*.

• *Constructions*
Starting with an object X, systematically produce new objects Y with $\mathcal{D}_Y \simeq \mathcal{D}_X$.

Motivating example – BGP reflections

A *BGP reflection* of a quiver is a new quiver obtained by inverting all arrows incident to a vertex which is a *sink* or a *source*.

Example.



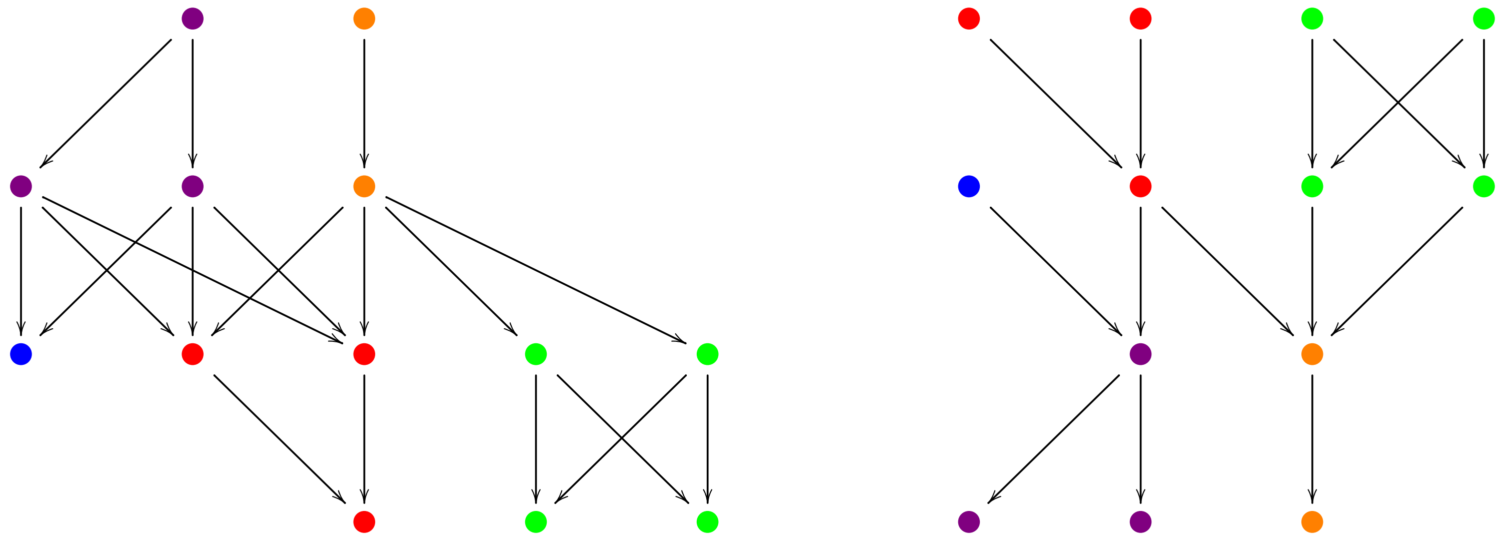
Theorem. [Bernstein-Gelfand-Ponomarev 1973, Happel 1981]
Let Q and Q' be two quivers without oriented cycles.
Then the path algebras KQ and KQ' are *derived equivalent* if and only if Q' can be obtained from Q by a *sequence of BGP reflections*.

Goals and Aims

How are the combinatorial properties of an object reflected in the homological and representation-theoretic properties of the associated derived category?

- *Algorithm* to decide on derived equivalence?
- *Dependence* on the auxiliary algebraic data?
- Basic *moves*?
- Enough *invariants*?
- Complete *description* of derived equivalence classes?

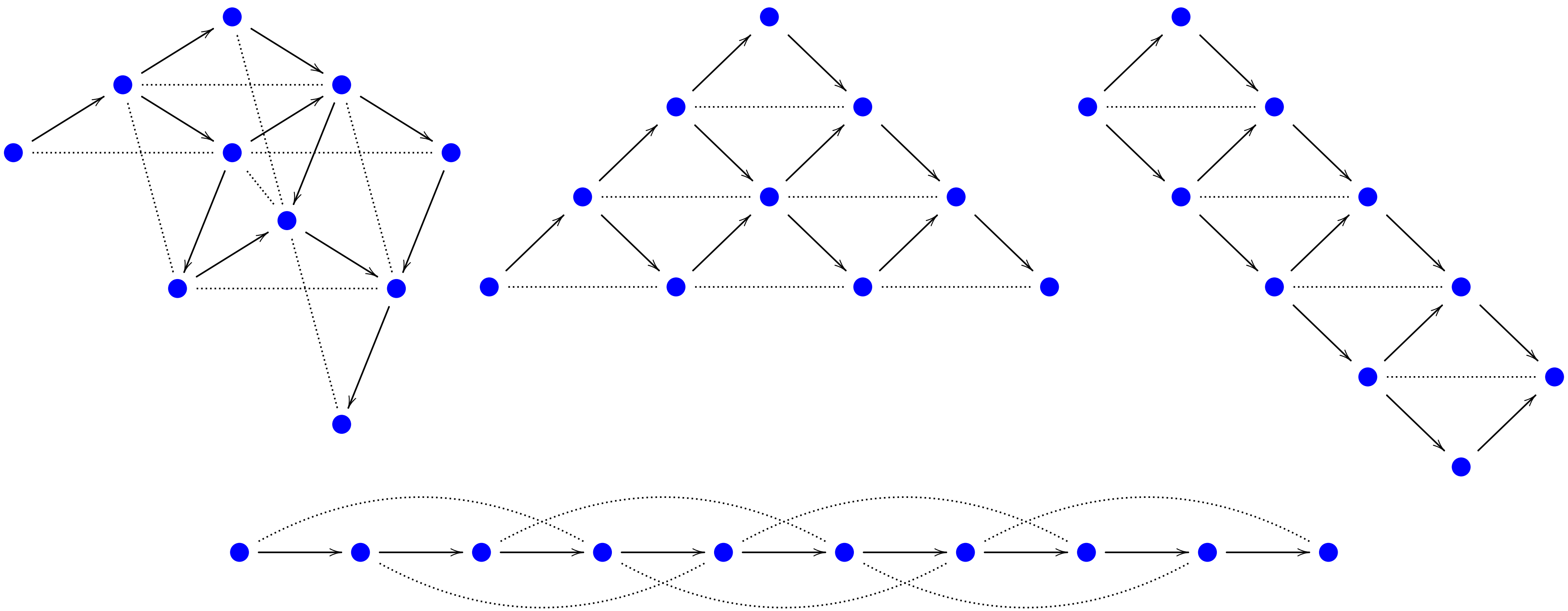
Construction 1 – Bipartite



Ladkani, *On derived equivalences of categories of sheaves over finite posets*, JPAA, 2008.

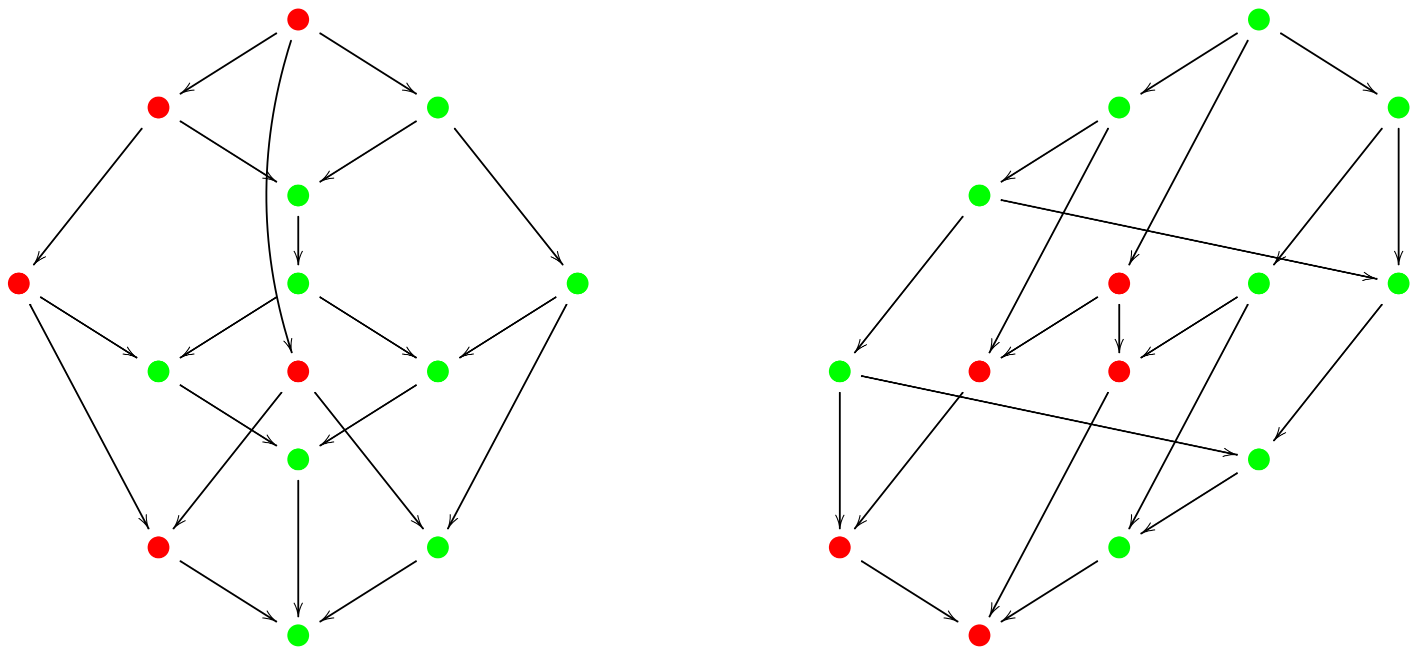
Ladkani, *Derived equivalences of triangular matrix rings arising from extensions of tilting modules*, Algebr. Represent. Theory, 2011.

Construction 2 – Tensor



Ladkani, *On derived equivalences of lines, rectangles and triangles*, J. London Math. Soc., 2013

Construction 3 – Posets



Ladkani, *Universal derived equivalences of posets*, `arXiv:0705.0946`
Universal derived equivalences of posets of tilting objects, `arXiv:0708.1287`
Universal derived equivalences of posets of cluster tilting objects, `arXiv:0710.2860`

Quivers from surface triangulations

A *marked bordered surface* is a pair (S, M) consisting of:

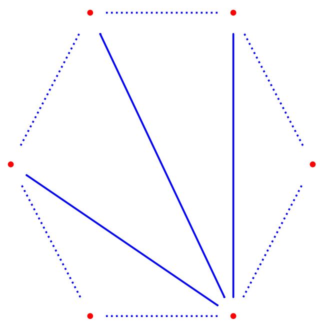
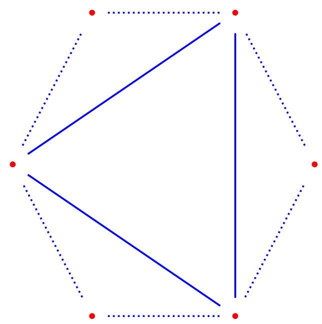
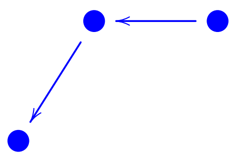
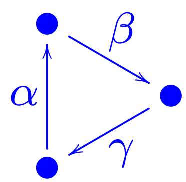
- a compact, connected, oriented surface S (possibly with boundary),
- a finite set $M \subset S$ of *marked points*, containing at least one point on each boundary component of S.

(S, M) is *unpunctured* if $M \subset \partial S$.

Facts. [Fomin-Shapiro-Thurston 2008]

triangulation	$\rightsquigarrow$	*adjacency quiver*
flip	$\rightsquigarrow$	mutation
all triangulations of (S, M)	$\rightsquigarrow$	finite mutation class

Example – Triangulations of a disc

Triangulation		
Adjacency quiver		
Potential	0	$\alpha\beta\gamma$
Algebra	KQ	$KQ/(\alpha\beta, \beta\gamma, \gamma\alpha)$

Derived equivalences for surface triangulations

(Local picture)

Consider marked *unpunctured* surfaces.

Proposition [L]. For a single flip, TFAE:

(i) The *number of inner triangles* is preserved;

(ii) The *number of arrows* in the adjacency quivers is preserved;

(iii) The corresponding algebras are *derived equivalent*.

Call such flips *“good” flips*.

Derived equivalences for surface triangulations

(Global picture)

Theorem [L]. Two algebras arising from surface triangulations are *derived equivalent* if and only if they are connected by a *sequence of good flips*.

Corollary. For these algebras there is an effectively computable *complete derived invariant*, hence the question of derived equivalence is decidable.

Theorem [L]. When the surface has just *one marked point* on each boundary component, the algebras arising from its triangulations form a *complete derived equivalence class* of algebras.