“I predict a new subject of statistical topology. Rather than count the number of holes, Betti numbers, etc., one will be more interested in the distribution of such objects on noncompact manifolds as one goes out to infinity.” — Isadore Singer, 2004.
Some earlier work:
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- We need to put topological data analysis on firmer probabilistic foundations in order to quantify inference, etc.
- Certain situations in physics seem to be well modeled by probabilistic topology.
- Why do so many groups / manifolds / simplicial complexes / etc. seem to have a certain topological property?

E.g. many simplicial complexes and posets arising in combinatorics are homotopy equivalent to bouquets of spheres. But why does this happen so often?
The probabilistic method provides existence proofs.
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- Ramsey theory and extremal graph theory e.g. Erdős, ...
- Geometric group theory — e.g. Gromov, Žuk
- Expander graphs — e.g. Pinsker, Barzdin & Kolmogorov, etc.
Random graphs
Define $G(n, p)$ to be the probability space of graphs on vertex set $[n] = \{1, 2, \ldots, n\}$, where each edge has probability $p$, independently.
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It is often useful to think of a growth process associated with $G(n, p)$ where edges are added one at a time.
Theorem (Erdős–Rényi, 1959)

Let $\epsilon > 0$ be fixed and $G \sim G(n, p)$. Then

$$
P[G \text{ is connected}] \rightarrow \begin{cases} 
1 & : p \geq (1 + \epsilon) \log n/n \\
0 & : p \leq (1 - \epsilon) \log n/n
\end{cases}
$$

They actually proved a slightly sharper result.
Theorem (Erdős–Rényi, 1959)

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Theorem (Erdős–Rényi, 1959)

Let $c \in \mathbb{R}$ be fixed and $G \sim G(n, p)$. If

$$p = \frac{\log n + c}{n},$$

then $\tilde{\beta}_0(G)$ is asymptotically Poisson distributed with mean $e^{-c}$ and in particular

$$\mathbb{P}[G \text{ is connected}] \to e^{-e^{-c}}$$

as $n \to \infty$. 
The first step is to show that if \( p \approx \log n/n \), then the probability that there are any components of order \( i \), with \( 2 \leq i \leq n/2 \) tends to 0 as \( n \to \infty \).
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A “union bound” argument shows that it is sufficient to show that

$$
\sum_{k=2}^{\lfloor n/2 \rfloor} \binom{n}{k} k^{k-2} p^{k-1} (1 - p)^{k(n-k)} \rightarrow 0,
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The first step is to show that if \( p \approx \log n / n \), then the probability that there are any components of order \( i \), with \( 2 \leq i \leq n/2 \) tends to 0 as \( n \to \infty \).

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\]

as \( n \to \infty \).

So w.h.p. \( G(n, p) \) consists of a giant component and isolated vertices.
Now set \( p = (\log n + c)/n \), where \( c \in \mathbb{R} \) is fixed. By linearity of expectation, the expected number of isolated vertices \( V \) is

\[
\mathbb{E}[V] = n(1 - p)^{n-1},
\]

and since \( 1 - p \approx e^{-p} \) for \( p \approx 0 \), we have

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Now set $p = \frac{\log n + c}{n}$, where $c \in \mathbb{R}$ is fixed. By linearity of expectation, the expected number of isolated vertices $V$ is

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as $n \to \infty$.

By computing the higher moments, one can show that $V$ approaches a Poisson distribution with mean $e^{-c}$. 
Comments:

The proof is phrased in terms of cohomology rather than in terms of homology.

The ultimate obstruction to connectivity is isolated vertices.

Once $p$ is a little bit past the connectivity threshold, $G \cong G(n, p)$ is an expander.
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- The ultimate obstruction to connectivity is isolated vertices.
- Once $p$ is a little bit past the connectivity threshold, $G \sim G(n, p)$ is an *expander*.
Theorem (Hoffman, K., Paquette, 2012)

Fix $k \geq 0$ and $\epsilon > 0$ be fixed. Let $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$ be the eigenvalues of the normalized Laplacian of the random graph $G(n, p)$. There is a constant $C_k$ so that when

$$p \geq \frac{(k + 1) \log n + C_k \sqrt{\log n \log \log n}}{n}$$

is satisfied, then

$$\lambda_2 > 1 - \epsilon,$$

with probability at least $1 - o(n^{-k})$. 
It is worth noting that there is another topological phase transition for $G(n, p)$, namely when cycles first appear.
Theorem (Pittel)

Let $c > 0$ be fixed, $p = c/n$, and $G \sim G(n, p)$. Then

$$\mathbb{P}[H_1(G) = 0] \rightarrow \begin{cases} 0 & : c \geq 1 \\ \frac{\sqrt{1-c}}{\exp(c/2+c^2/4)} & : c < 1 \end{cases}$$

as $n \rightarrow \infty$. 

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Random flag complexes
Let $X \sim X(n, p)$ be the clique complex (or flag complex) of $G \sim G(n, p)$, i.e. the maximal simplicial complex compatible with $G$. 
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Note: every simplicial complex is homeomorphic to a flag complex, e.g. by barycentric subdivision, so $X(n, p)$ puts a measure on a wide range of topologies.
Betti numbers of random flag complexes on $n = 100$ vertices, with $0 \leq p \leq 0.6$. 
Theorem (K., 2007)

*Fix* $k \geq 1$, and let $X \sim X(n, p)$. If

$$\frac{1}{n^{1/k}} \ll p \ll \frac{1}{n^{1/(k+1)}}$$

*then*

$$\mathbb{P}[H_k(X) = 0] \rightarrow 0$$

*as* $n \rightarrow \infty$. 
Theorem (K., 2007)

Fix $k \geq 1$, and let $X \sim X(n, p)$.

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Theorem (K., 2007)

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  \]
  as $n \to \infty$.

- Also, if
  \[
  p \gg \frac{1}{n^{1/(2k+1)}}
  \]
  then
  \[
  \mathbb{P}[H_k(X) = 0] \to 1
  \]
  as $n \to \infty$. 
In fact, much more can be said about the size of homology in this regime. The limiting expectation has a nice formula.
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**Theorem (K., 2007)**

If

\[
\frac{1}{n^{1/k}} \ll p \ll \frac{1}{n^{1/(k+1)}},
\]

then

\[
\frac{\mathbb{E}[\beta_k]}{\binom{n}{k+1} p^{(k+1)/2}} \rightarrow 1
\]

as \( n \rightarrow \infty \).
Moreover, the $k$th Betti number satisfies a central limit theorem.
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**Theorem (K.–Meckes, 2010))**

If

\[
\frac{1}{n^{1/k}} \ll p \ll \frac{1}{n^{1/(k+1)}}
\]

then

\[
\frac{\beta_k - \mathbb{E}[\beta_k]}{\sqrt{\text{Var} [\beta_k]}} \to \mathcal{N}(0, 1)
\]

as $n \to \infty$. 
The following new result gives a sharp vanishing threshold for cohomology.

\[ P[H_k(Y, Q) = 0] > \frac{1}{p} \begin{cases} \log \frac{n}{n} & \text{if } 0 < \epsilon < 1 \\ \frac{1}{k+1} & \text{if } n^{1/k} \approx p \approx \frac{k}{2} \log \frac{1}{\epsilon} \end{cases} \]

This provides a generalization of the Erdős–Rényi Theorem.
The following new result gives a sharp vanishing threshold for cohomology.

**Theorem (K., 2012)**

Let $0 < \varepsilon < 1$ and $k$ be fixed and $X \sim X(n, p)$. Then

$$\mathbb{P}[\mathcal{H}^k(Y, \mathbb{Q}) = 0] \rightarrow \begin{cases} 1 & : p \geq \left( \frac{(k/2+1+\varepsilon) \log n}{n} \right)^{1/(k+1)} \\ 0 & : \frac{1}{n^{1/k}} \ll p \leq \left( \frac{(k/2+1-\varepsilon) \log n}{n} \right)^{1/(k+1)} \end{cases}$$

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Besides the earlier spectral gap theorem, the main tool is the following.
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**Theorem**

(Garland, 1973, Ballman–Świątkowski, 1997) If $\Delta$ is a pure $k$-dimensional simplicial complex, such that the link $lk_\Delta(\sigma)$ of every $(k - 2)$-face $\sigma$ is connected and has spectral gap satisfying

$$\lambda_2[lk_\Delta(\sigma)] > 1 - 1/k,$$

then $H^{k-1}(\Delta, \mathbb{Q}) = 0$. 

We have the following corollary.

**Corollary**

*Fix* $d \geq 0$, and let $X \sim X(n, p)$ be a random flag complex, where

$$\frac{1}{n^2/d} \ll p \ll \frac{1}{n^2/(d+1)}.$$

Then w.h.p. $X$ is $d$-dimensional and $\tilde{H}_i(X, \mathbb{Q}) = 0$ unless $i = \lfloor d/2 \rfloor$. 
We have the following corollary.

**Corollary**

*Fix $d \geq 0$, and let $X \sim X(n, p)$ be a random flag complex, where

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Then w.h.p. $X$ is $d$-dimensional and $\tilde{H}_i(X, \mathbb{Q}) = 0$ unless $i = \lfloor d/2 \rfloor$.

Moreover, if $d \geq 6$ then w.h.p. $X$ is rationally homotopy equivalent to a wedge of $\lfloor d/2 \rfloor$-spheres.
One consequence is that now we can predict Betti numbers of random flag complexes fairly well, even for small $n...$
The predicted Betti numbers are given by the absolute Euler characteristic.

\[ |\mathbb{E}[\chi]| = \left| \binom{100}{1} - \binom{100}{2} p + \binom{100}{3} p^3 - \ldots \right| . \]
The actual Betti numbers.
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The new results also say something with respect to *persistent homology* of the random filtration associated with $X(n, p)$. 
Corollary

Fix $\epsilon > 0$ and $k \geq 1$. Let $b, d$ be the birth time and death time, respectively of the longest bar in persistent $H_k$. Then w.h.p.

$$\frac{\log d - \log b}{\log n} \approx \frac{1}{k(k + 1)}.$$
Open problems
It might be possible to slightly sharpen the main result.
Conjecture

If

\[ p = \left( \frac{(k/2 + 1) \log n + (k/2) \log \log n + c}{n} \right)^{1/(k+1)}, \]

where \( c \in \mathbb{R} \) is constant, then the dimension of \( k \)th cohomology \( \beta^k \) approaches a Poisson distribution with mean

\[ \mu = \frac{(k/2 + 1)^{k/2}}{(k + 1)!} e^{-c}. \]

In particular,

\[ \mathbb{P}[H^k(X, \mathbb{Q}) = 0] \to \exp \left[ -\frac{(k/2 + 1)^{k/2}}{(k + 1)!} e^{-c} \right], \]

as \( n \to \infty. \)
How to handle torsion in homology of random complexes?
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There exists a 2-dimensional $\mathbb{Q}$-acyclic simplicial complex $S$ on 31 vertices with

$$|H_1(S, \mathbb{Z})| = 736712186612810774591.$$
How to handle torsion in homology of random complexes?

There exists a 2-dimensional \( \mathbb{Q} \)-acyclic simplicial complex \( S \) on 31 vertices with

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|H_1(S, \mathbb{Z})| = 736712186612810774591.
\]

Gil Kalai showed that, on average, \( \mathbb{Q} \)-acyclic complexes have enormous torsion in homology.
Still, I conjecture the following.
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**Conjecture**

If $X \sim X(n, p)$ is a random flag complex with

$$\frac{1}{n^2/d} \ll p \ll \frac{1}{n^2/(d+1)},$$

where $d \geq 6$ is fixed, then w.h.p. $X$ is homotopy equivalent to a wedge of $\lfloor d/2 \rfloor$-spheres.
Still, I conjecture the following.

**Conjecture**

If $X \sim X(n, p)$ is a random flag complex with

$$\frac{1}{n^2/d} \ll p \ll \frac{1}{n^2/(d+1)},$$

where $d \geq 6$ is fixed, then w.h.p. $X$ is homotopy equivalent to a wedge of $\lceil d/2 \rceil$-spheres.

By uniqueness of Moore spaces, this is equivalent to showing that $H_*(X)$ is torsion-free.
Thanks for your time and attention!
Acknowledgements

Collaborators: Chris Hoffman, Elizabeth Meckes, Elliot Paquette


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M. Kahle, Topology of random clique complexes (Discrete Math. 309 (2009), no. 6, 1658–1671.)