On topological abstraction of higher dimensional automata

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Higher dimensional automata

A *higher dimensional automaton* over a monoid M (M-HDA) is a tuple

$$\mathcal{A} = (P, I, F, \lambda)$$

where

- P is a precubical set,
- $I \subseteq P_0$ is a set of *initial states*,
- $F \subseteq P_0$ is a set of *final states*,
- $\lambda \colon P_1 \to M$ is a map, called the *labelling function*, such that

$$\lambda(d_i^0 x) = \lambda(d_i^1 x)$$

  for all $x \in P_2$ and $i \in \{1, 2\}$.

Higher dimensional automata

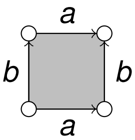


Figure: Cubes represent independence of actions

Paths

Let k and l be integers such that $k \leq l$. The *precubical interval* $[\![k, l]\!]$ is the precubical set

$$\overset{k}{\bullet} \to \overset{k+1}{\bullet} \to \cdots \to \overset{l-1}{\bullet} \to \overset{l}{\bullet}\,.$$

A *path of length* k in a precubical set P is a morphism of precubical sets $\omega\colon [\![0, k]\!] \to P$.

The set of paths in P is denoted by $P^{\mathbb{I}}$.

Remark

A path of length $k \geq 1$ can be identified with a sequence $(x_1, \ldots, x_k)$ of elements of P_1 such that $d_1^0 x_{j+1} = d_1^1 x_j$ $(1 \leq j < k)$.

The language accepted by an HDA

The *extended labelling function* of an an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is the map

$$\overline{\lambda} \colon P^{\mathbb{I}} \to M$$

defined by

$$\overline{\lambda}(x_1, \ldots, x_k) = \lambda(x_1) \cdots \lambda(x_k).$$

If ω is a path of length 0, then we set

$$\overline{\lambda}(\omega) = 1.$$

The *language accepted by* $\mathcal{A}$ is the set

$$L(\mathcal{A}) = \{\overline{\lambda}(\omega) : \omega \in P^{\mathbb{I}},\ \omega(0) \in I,\ \omega(\mathrm{length}(\omega)) \in F\}.$$

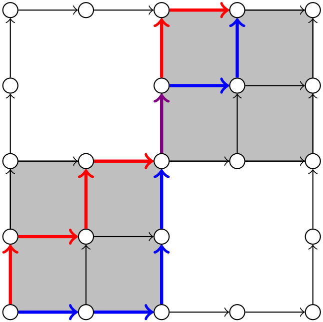
Dihomotopy

Two paths ω and ν in a precubical set P are said to be *elementarily dihomotopic* if there exist paths $\alpha, \beta \in P^{\mathbb{I}}$ and an element $z \in P_2$ such that

- $d_1^0 d_1^0 z = \alpha(\mathrm{length}(\alpha))$, $d_1^1 d_1^1 z = \beta(0)$,
- $\{\omega, \nu\} = \{\alpha \cdot (d_1^0 z, d_2^1 z) \cdot \beta, \alpha \cdot (d_2^0 z, d_1^1 z) \cdot \beta\}$.

Dihomotopy is the equivalence relation generated by elementary dihomotopy.

Dihomotopy



Dihomotopic paths

The trace language of an HDA

Let $\mathcal{A} = (P, I, F, \lambda)$ be an M-HDA. We say that $a, b \in M$ are *independent in* $\mathcal{A}$ if

1. $a, b \in \lambda(P_1)$;
2. $a \neq b$;
3. for all paths $\omega \in P^{\mathbb{I}}$ and elements $u, v \in M$ with $\overline{\lambda}(\omega) \in \{uabv, ubav\}$ there exists a path $\nu \in P^{\mathbb{I}}$ such that ω and ν are dihomotopic and $\{\overline{\lambda}(\omega), \overline{\lambda}(\nu)\} = \{uabv, ubav\}$.

We denote by $\equiv_{\mathcal{A}}$ the smallest congruence relation in M such that ab and ba are congruent for all independent elements $a, b \in M$.

The quotient monoid $M/ \equiv_{\mathcal{A}}$ is called the *trace monoid* of $\mathcal{A}$, and the canonical projection $M \to M/ \equiv_{\mathcal{A}}$ is denoted by $tr_{\mathcal{A}}$. The *trace language* of $\mathcal{A}$ is the set

$$TL(\mathcal{A}) = tr_{\mathcal{A}}(L(\mathcal{A})) \subseteq M/ \equiv_{\mathcal{A}} .$$

Stable and deterministic HDAs

We say that an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is *stable* if two elements $a, b \in M$ are independent if there exists an element $z \in P_2$ such that $\{\lambda(d_1^0 z), \lambda(d_2^0 z)\} = \{a, b\}$.

We say that an M-HDA is *deterministic* if it has exactly one initial state and if any two paths with the same starting point and the same label are equal.

Proposition

Let $\mathcal{A}$ be a deterministic and stable M-HDA. Then two paths with the same starting point are dihomotopic if and only if they have congruent labels.

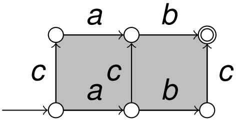
The trace category of an HDA

The *fundamental category* of a precubical set P is the category $\vec{\pi}_1(P)$ whose objects are the vertices of P and whose morphisms are the dihomotopy classes of paths in P.

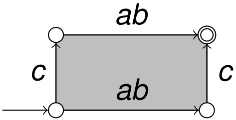
A vertex v of a precubical set P is said to be *maximal (minimal)* if there is no element $x \in P_1$ such that $d_1^0 x = v$ ($d_1^1 x = v$). The sets of maximal and minimal elements of P are denoted by $M(P)$ and $m(P)$ respectively.

The *trace category* of an M-HDA $\mathcal{A} = (P, I, F, \lambda)$, $TC(\mathcal{A})$, is the full subcategory of $\vec{\pi}_1(P)$ generated by $I \cup F \cup m(P) \cup M(P)$.

Two HDAs



(a) $\mathcal{A}$



(b) $\mathcal{B}$

Figure: Two HDAs $\mathcal{A}$ and $\mathcal{B}$ over the free monoid on $\{a, b, c\}$

Tensor product

Given two precubical sets P and Q, the *tensor product* $P \otimes Q$ is the precubical set defined by

$$(P \otimes Q)_n = \coprod_{p+q=n} P_p \times Q_q.$$

and

$$d_i^k(x,y) = \begin{cases} (d_i^k x, y), & 1 \leq i \leq \deg(x), \\ (x, d_{i-\deg(x)}^k y), & \deg(x)+1 \leq i \leq \deg(x)+\deg(y). \end{cases}$$

Remark

$$|[\![0,k_1]\!] \otimes \cdots \otimes [\![0,k_n]\!]| = [0,k_1] \times \cdots \times [0,k_n].$$

Weak morphisms

A *weak morphism* from a precubical set P to a precubical set Q is a continuous map $f\colon |P| \to |Q|$ such that the following two conditions hold:

1. f sends vertices to vertices;
2. for all integers $n, k_1, \ldots, k_n \geq 1$ and every morphism of precubical sets $\xi\colon [\![0, k_1]\!] \otimes \cdots \otimes [\![0, k_n]\!] \to P$ there exist integers $l_1, \ldots, l_n \geq 1$, a morphism of precubical sets

$$\chi\colon [\![0, l_1]\!] \otimes \cdots \otimes [\![0, l_n]\!] \to Q$$

and a homeomorphism

$$\begin{aligned}\phi\colon |[\![0, k_1]\!] \otimes \cdots \otimes [\![0, k_n]\!]| &= [0, k_1] \times \cdots \times [0, k_n] \\ \to \quad |[\![0, l_1]\!] \otimes \cdots \otimes [\![0, l_n]\!]| &= [0, l_1] \times \cdots \times [0, l_n],\end{aligned}$$

such that $f \circ |\xi| = |\chi| \circ \phi$ and ϕ is a dihomeomorphism, i.e. ϕ and ϕ^{-1} preserve the natural partial order of $\mathbb{R}^n$.

Weak morphisms

Let $f\colon |P| \to |Q|$ be a weak morphism of precubical sets and $\omega\colon [\![0,k]\!] \to P$ ($k \geq 0$) be a path. We denote by $f^{\mathbb{I}}(\omega)$ the unique path $\nu\colon [\![0,l]\!] \to Q$ for which there exists a dihomeomorphism $\phi\colon |[\![0,k]\!]| = [0,k] \to |[\![0,l]\!]| = [0,l]$ such that $f \circ |\omega| = |\nu| \circ \phi$.

A *weak morphism* from an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ to an M-HDA $\mathcal{B} = (Q, J, G, \mu)$ is a weak morphism $f\colon |P| \to |Q|$ such that $f(I) \subseteq J$, $f(F) \subseteq G$ and $\overline{\mu} \circ f^{\mathbb{I}} = \overline{\lambda}$.

Proposition

Weak morphisms preserve dihomotopy. Consequently, if f is a weak morphism from an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ to an M-HDA $\mathcal{B} = (Q, J, G, \mu)$ such that $f(m(P)) \subseteq m(Q)$ and $f(M(P)) \subseteq M(Q)$, then f induces a functor $f_*\colon TC(\mathcal{A}) \to TC(\mathcal{B})$.

Homeomorphic abstraction

Consider two M-HDAs $\mathcal{A} = (P, I, F, \lambda)$ and $\mathcal{B} = (P', I', F', \lambda')$. We say that $\mathcal{A}$ is a *homeomorphic abstraction* of $\mathcal{B}$, or that $\mathcal{B}$ is a *homeomorphic refinement* of $\mathcal{A}$, if there exists a weak morphism f from $\mathcal{A}$ to $\mathcal{B}$ that is a homeomorphism and satisfies $f(I) = I'$ and $f(F) = F'$. We use the notation $\mathcal{A} \overset{\approx}{\to} \mathcal{B}$ to indicate that $\mathcal{A}$ is a homeomorphic abstraction of $\mathcal{B}$.

Remark

The relation $\overset{\approx}{\to}$ is a preorder on the class of M-HDAs.

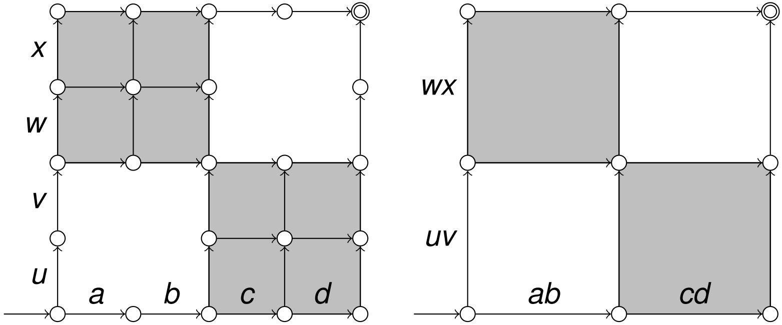
Homeomorphic abstraction

An M-HDA is said to be *weakly regular* if for every element x of degree 2, $d_1^0 x \neq d_2^0 x$ and $d_1^1 x \neq d_2^1 x$.

Theorem

Suppose that $\mathcal{A} \overset{\approx}{\to} \mathcal{B}$. If $\mathcal{A}$ is weakly regular, then $TC(\mathcal{A}) \cong TC(\mathcal{B})$. If $\mathcal{A}$ is stable and $\mathcal{B}$ is stable and deterministic, then $TL(\mathcal{A}) \cong TL(\mathcal{B})$.

Homeomorphic abstraction



(a) An HDA (b) Homeomorphic abstraction

The homology graph

Let P be a precubical set. We say that a homology class $\alpha \in H(|P|)$ *points* to a homology class $\beta \in H(|P|)$ and write $\alpha \nearrow \beta$ if there exist precubical subsets $X, Y \subseteq P$ such that $\alpha \in \operatorname{im} H(|X| \hookrightarrow |P|)$, $\beta \in \operatorname{im} H(|Y| \hookrightarrow |P|)$ and for all $x \in X_0$ and $y \in Y_0$ there exists a path in P from x to y.

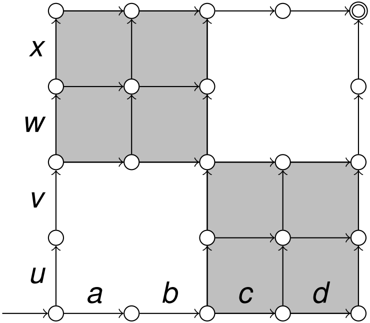
The *homology graph* of P is the directed graph whose vertices are the homology classes of $|P|$ and whose edges are given by the relation $\nearrow$.

The *homology graph* of an M-HDA $\mathcal{A} = (P, I, F, \lambda)$ is defined to be the homology graph of P.

Theorem

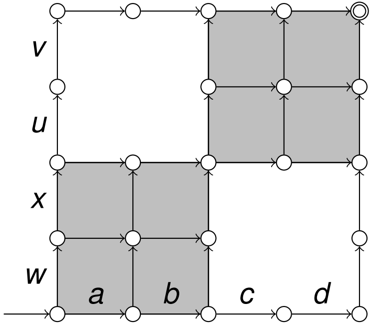
Let $f\colon |P| \to |Q|$ be a weak morphism of precubical sets that is a homeomorphism. Then for all homology classes $\alpha, \beta \in H(|P|)$, $\alpha \nearrow \beta$ if and only if $f_*(\alpha) \nearrow f_*(\beta)$.

Ordered holes



The homology class representing the lower hole points to the homology class representing the upper hole.

Unordered holes



The homology graph has no edges between non-zero classes of H_1.

Topological abstraction

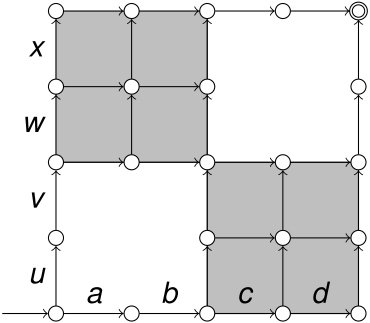
Consider two M-HDAs $\mathcal{A} = (P, I, F, \lambda)$ and $\mathcal{B} = (P', I', F', \lambda')$. We write $\mathcal{A} \overset{\sim}{\to} \mathcal{B}$ and say that $\mathcal{A}$ is a *topological abstraction* of $\mathcal{B}$, or that $\mathcal{B}$ is a *topological refinement* of $\mathcal{A}$, if there exists a weak morphism f from $\mathcal{A}$ to $\mathcal{B}$ such that

- $f(I) = I'$, $f(F) = F'$, $f(m(P)) = m(P')$, $f(M(P)) = M(P')$,
- $f_* \colon TC(\mathcal{A}) \to TC(\mathcal{B})$ is an isomorphism,
- f is a homotopy equivalence,
- for all homology classes $\alpha, \beta \in H(|P|)$, $\alpha \nearrow \beta$ if and only if $f_*(\alpha) \nearrow f_*(\beta)$.

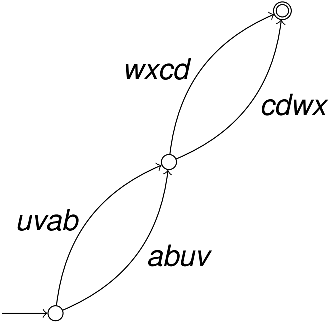
Theorem

Suppose that $\mathcal{A} \overset{\approx}{\to} \mathcal{B}$. If $\mathcal{A}$ is weakly regular, then $\mathcal{A} \overset{\sim}{\to} \mathcal{B}$. If $\mathcal{A}$ is stable and $\mathcal{B}$ is stable and deterministic, then a bijection $TL(\mathcal{A}) \to TL(\mathcal{B})$ is given by $tr_{\mathcal{A}}(l) \mapsto tr_{\mathcal{B}}(l)$.

Topological abstraction



(a) Ordered holes



(b) Topological abstraction