Geometry driven collapses for simplifying Čech complexes

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(*) Gipsa-lab **(**) Dassault Systèmes**

Workshop on Applied and Computational Algebraic Topology
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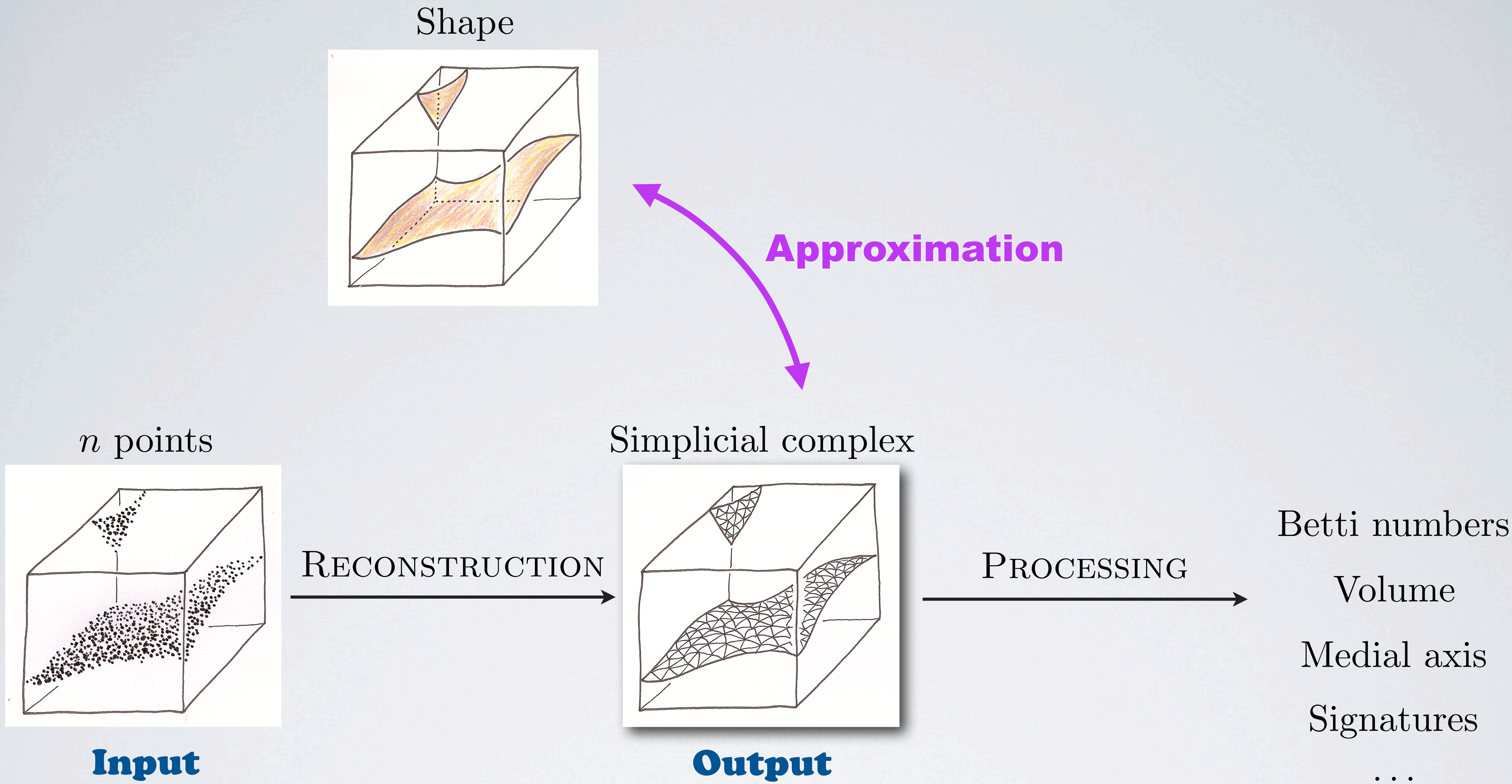
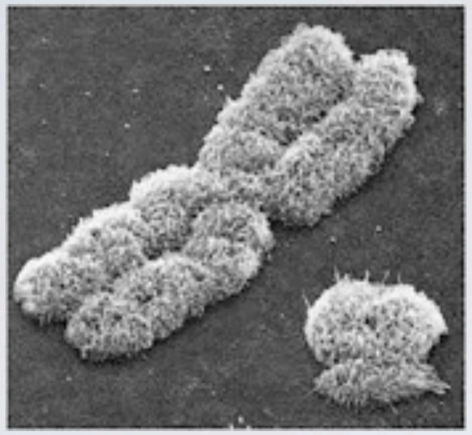
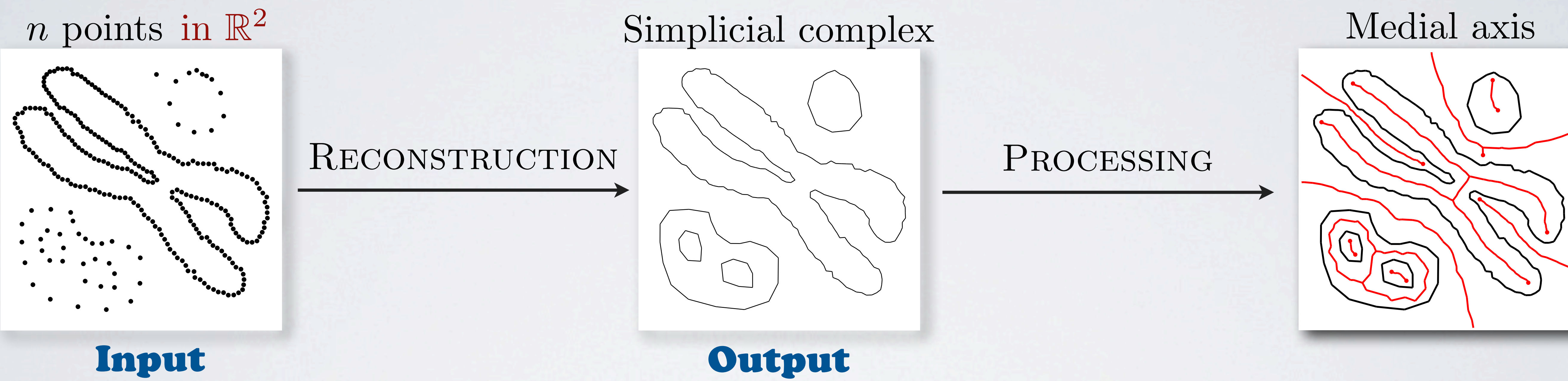
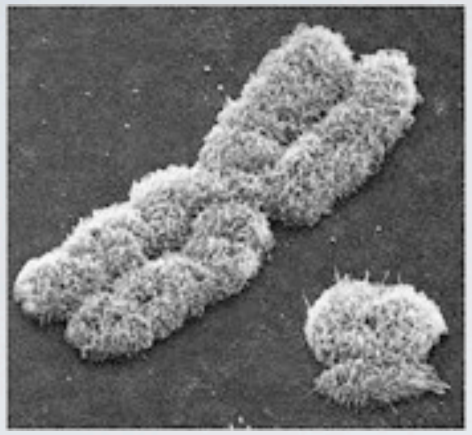

Shape
Approximation
n points
Simplicial complex
Reconstruction
Processing
Betti numbers
Volume
Medial axis
Signatures
...
Input
Output

in 2D

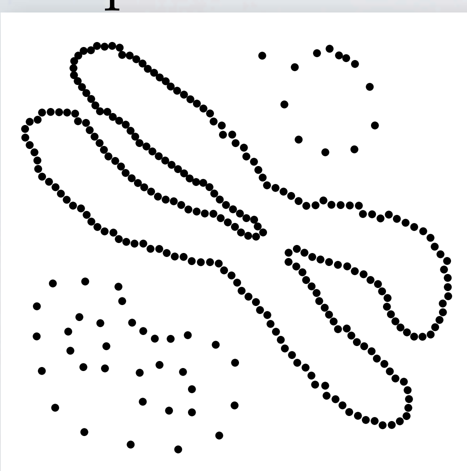



n points in $\mathbb{R}^2$
Input
Reconstruction
Simplicial complex
Output
Processing
Medial axis

in 2D

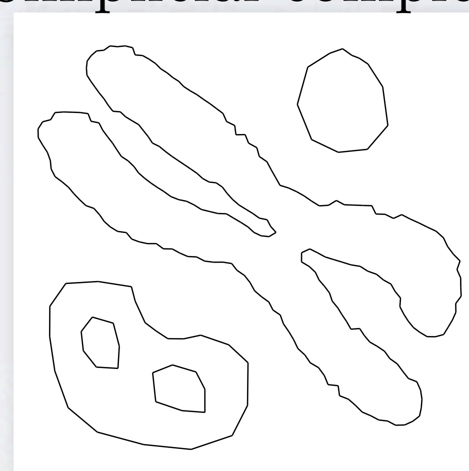


n points in $\mathbb{R}^2$



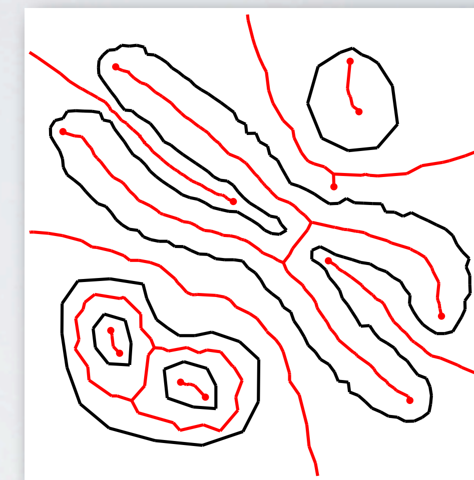
RECONSTRUCTION

Simplicial complex



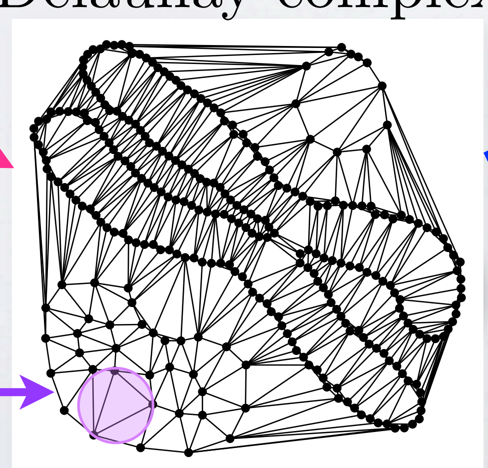
PROCESSING

Medial axis



BUILDING

Delaunay complex



Empty circle property

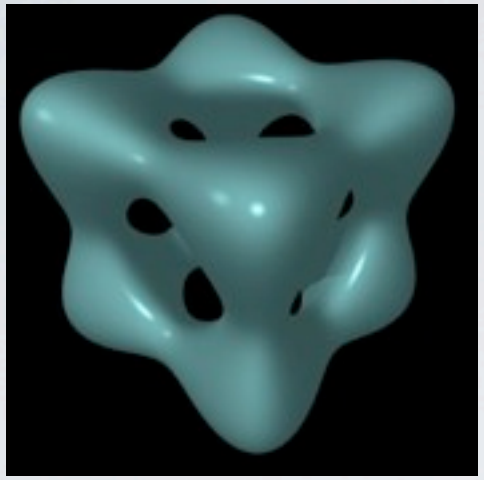
(1995 – 2005) HEURISTICS
(Crust, Power crust, Co-cone, Wrap, . . .)

✳ In $\mathbb{R}^2$, has size $O(n)$

CGAL

Delaunay of 10M points in 2D $\approx$ 10 s

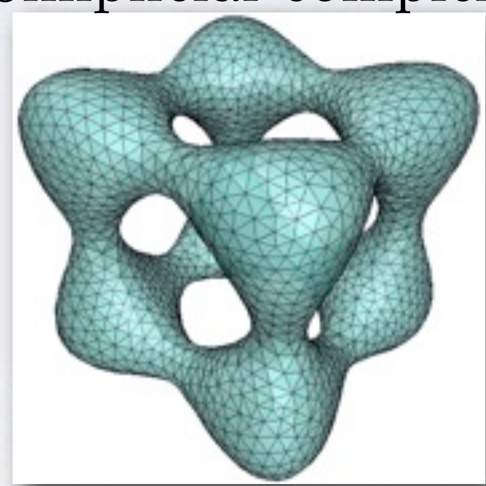
in 3D



n points in $\mathbb{R}^3$

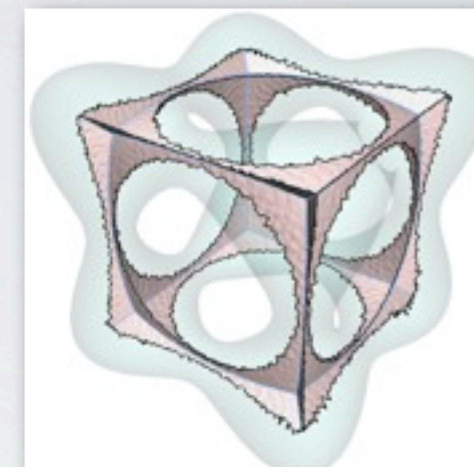
RECONSTRUCTION

Simplicial complex



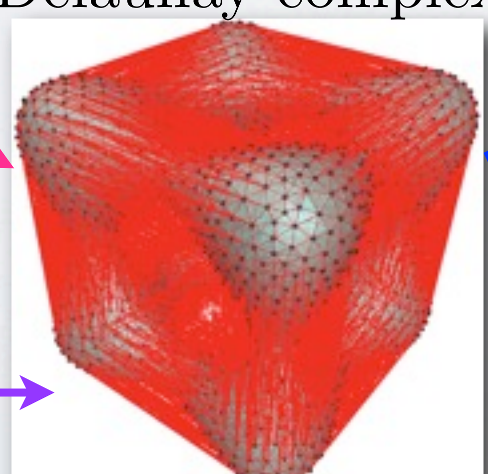
PROCESSING

Medial axis



BUILDING

Delaunay complex



Empty sheet property

(1995 – 2005)

(Crust, Power crust, Co-cone, Wrap, . . .)

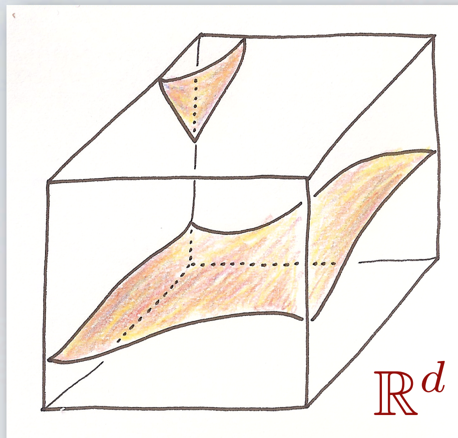
- In $\mathbb{R}^3$, has size $O(n^2)$
- In practice, has size $O(n)$

CGAL

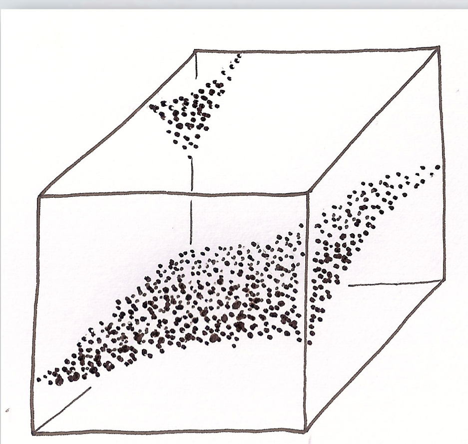
Delaunay of 10M points in 3D $\approx$ 80 s

in dD

Shape

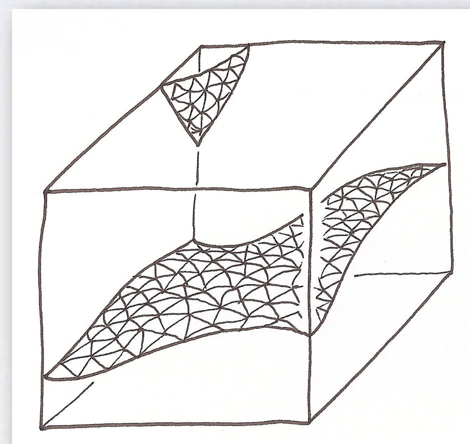


n points in $\mathbb{R}^d$



RECONSTRUCTION

Simplicial complex



PROCESSING

Betti numbers

Volume

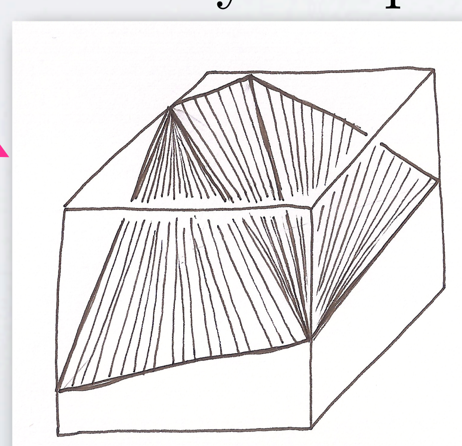
Medial axis

Signatures

...

BUILDING

Delaunay complex

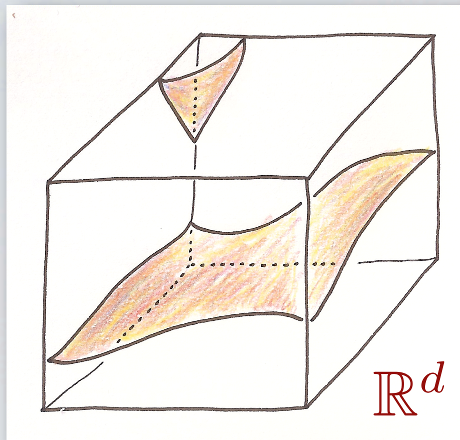


curse of dimensionality

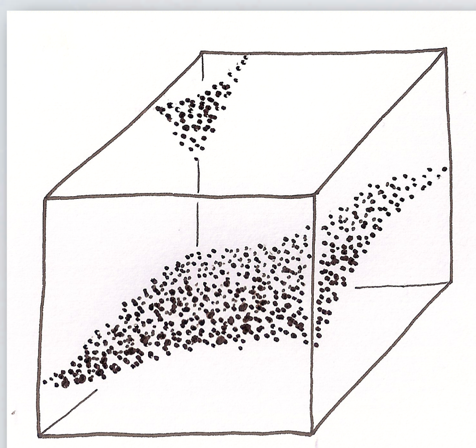
- ✳ In $\mathbb{R}^d$, has size $O(n^{\lceil d/2 \rceil})$
- ✳ The bound is tight (and achieved for points that sample curves).

in dD

Shape

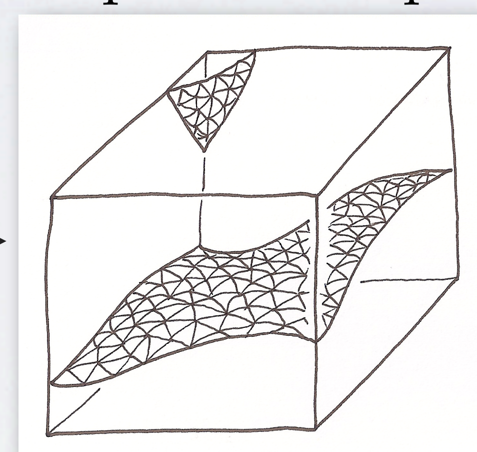


n points in $\mathbb{R}^d$



RECONSTRUCTION

Simplicial complex



PROCESSING

Betti numbers

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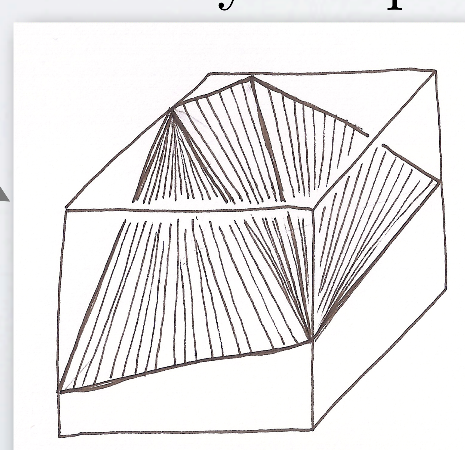
...

~~BUILDING~~

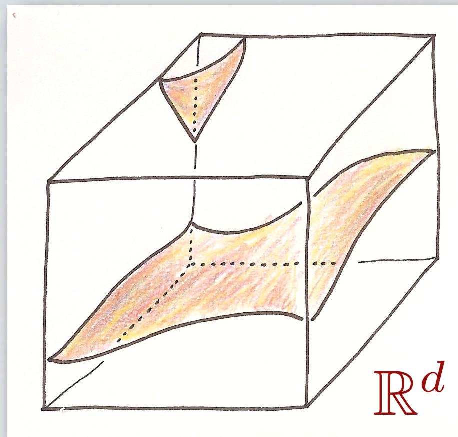
Delaunay complex



How to reconstruct without Delaunay?

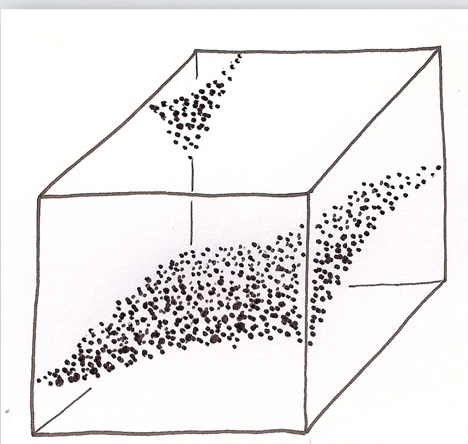
in dD

Shape



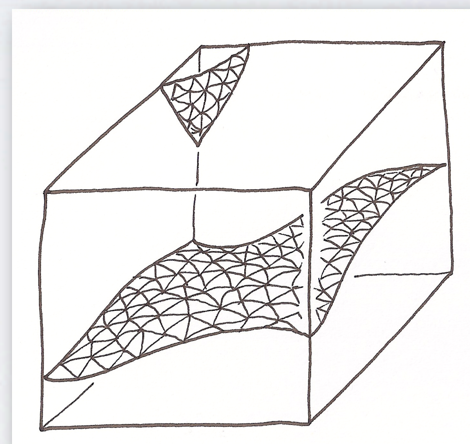
Guaranties on the result?

n points in $\mathbb{R}^d$



RECONSTRUCTION

Simplicial complex



PROCESSING

Betti numbers

Volume

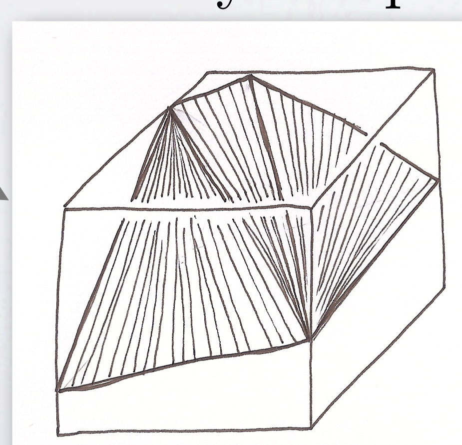
Medial axis

Signatures

…

~~BUILDING~~

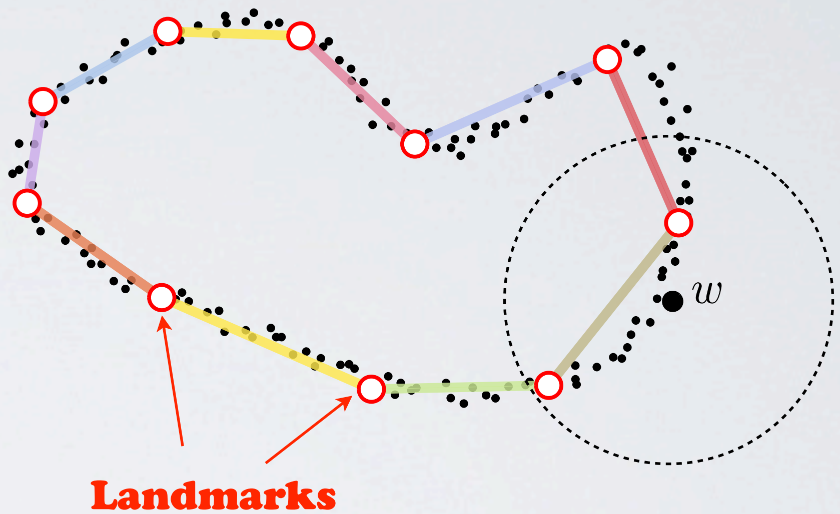
Delaunay complex



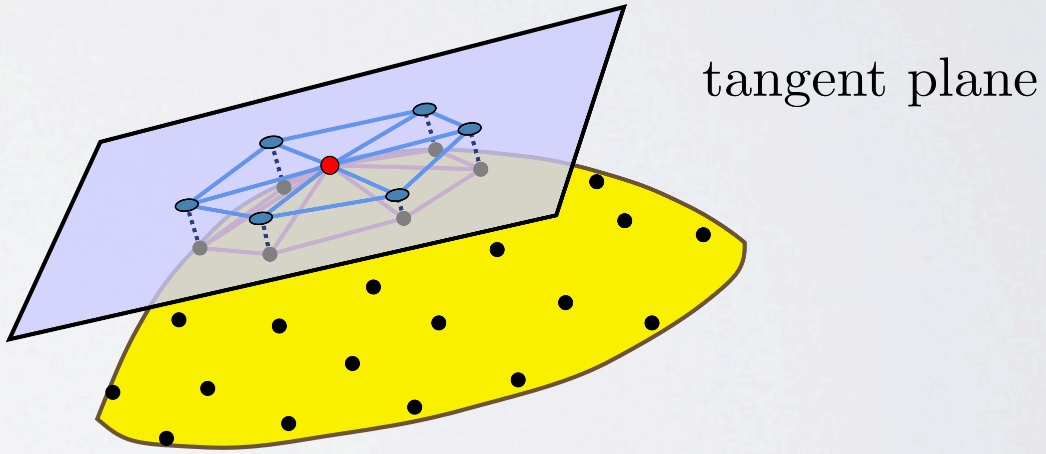
How to reconstruct without Delaunay?

How to reconstruct without building the whole Delaunay complex?

- weak Delaunay triangulation
  [V. de Silva 2008]

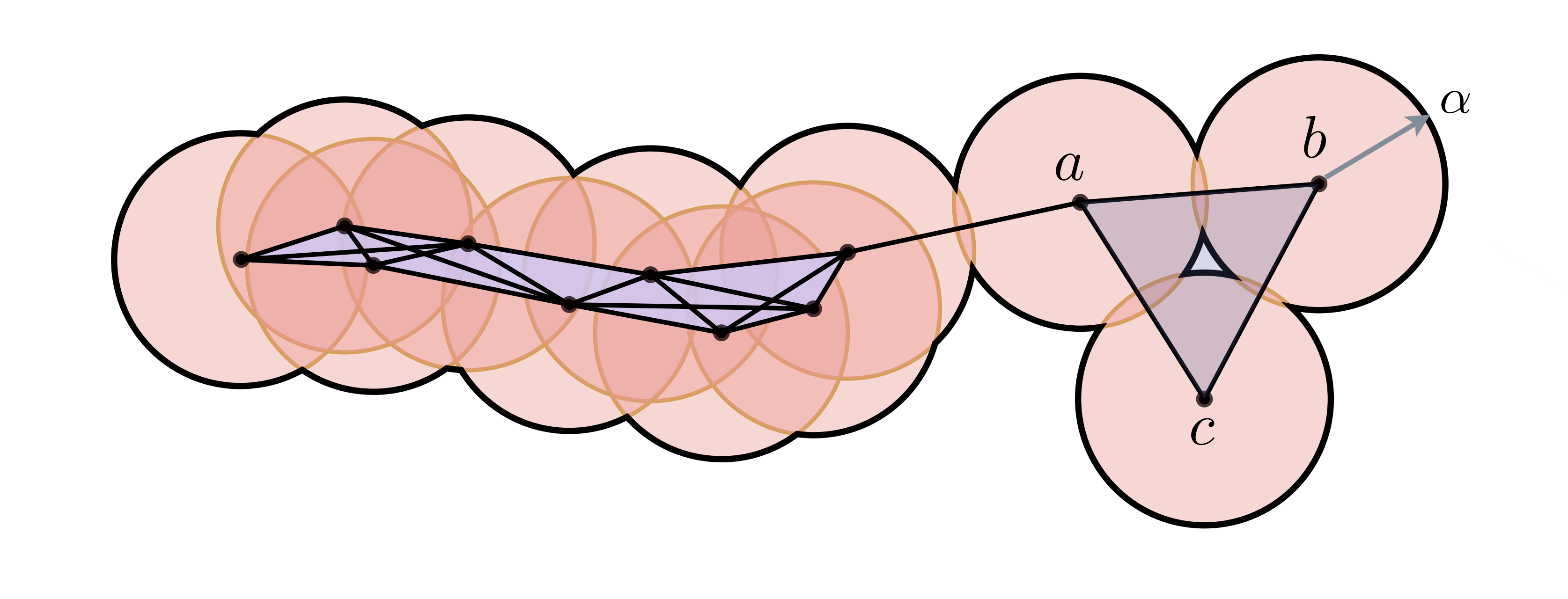


- tangential Delaunay complexes
  [J. D. Boissonnat & A. Ghosh 2010]



- Rips complexes
  our approach with André Lieutier and David Salinas

Rips complexes

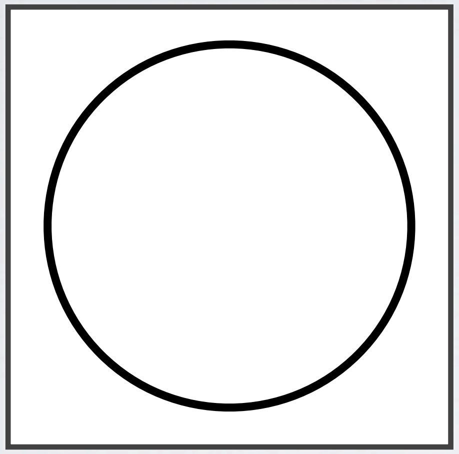


$$\mathrm{Rips}(P,\alpha) = \{\sigma \subset P \mid \mathrm{Diameter}(\sigma) \leq 2\alpha\}$$

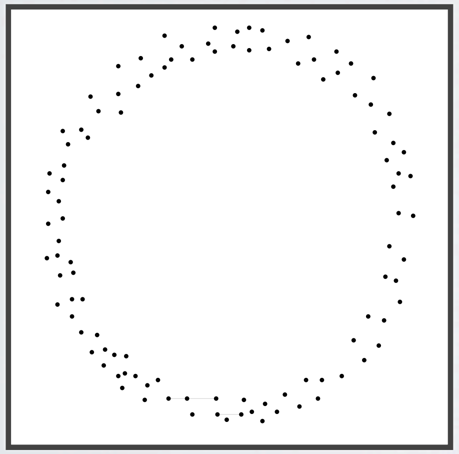
- **proximity graph G_α connects every pair of points within 2α**
- $\mathrm{Rips}(P,\alpha) = \mathrm{Flag}\, G_\alpha$ $\quad$ [Flag G = largest complex whose 1-skeleton is G]
- **compressed form of storage through the 1-skeleton**
- **easy to compute**

SHAPE RECONSTRUCTION

Shape A

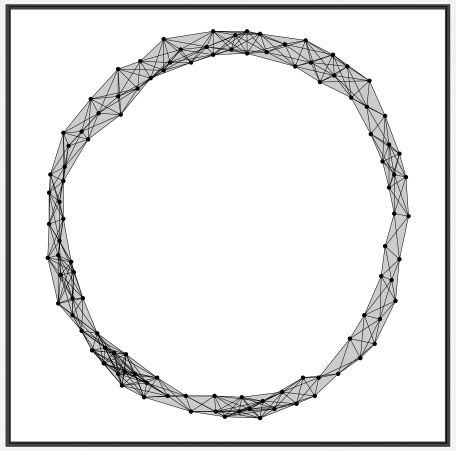


$P \subset \mathbb{R}^d$



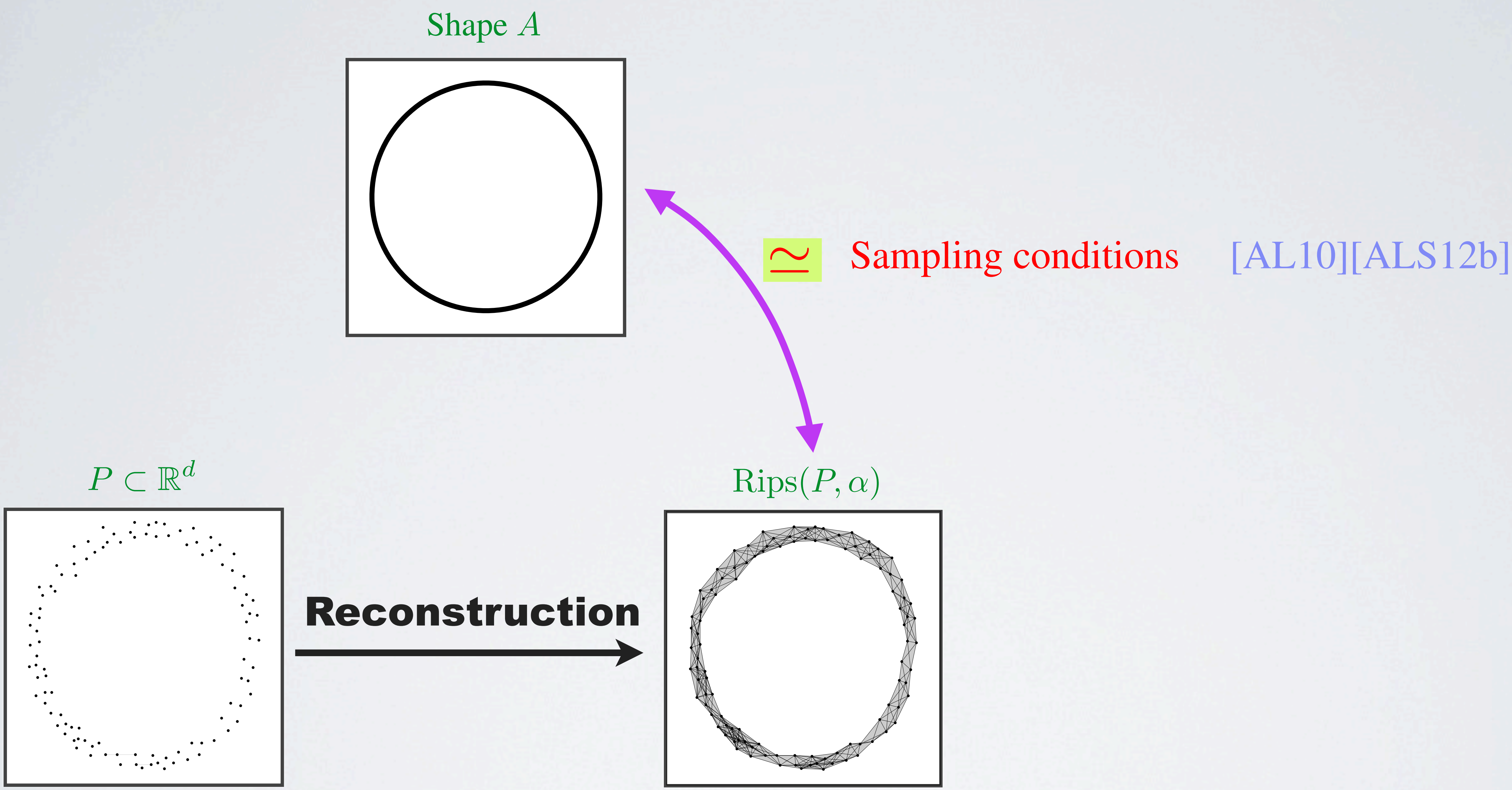
Reconstruction

$\mathrm{Rips}(P, \alpha)$



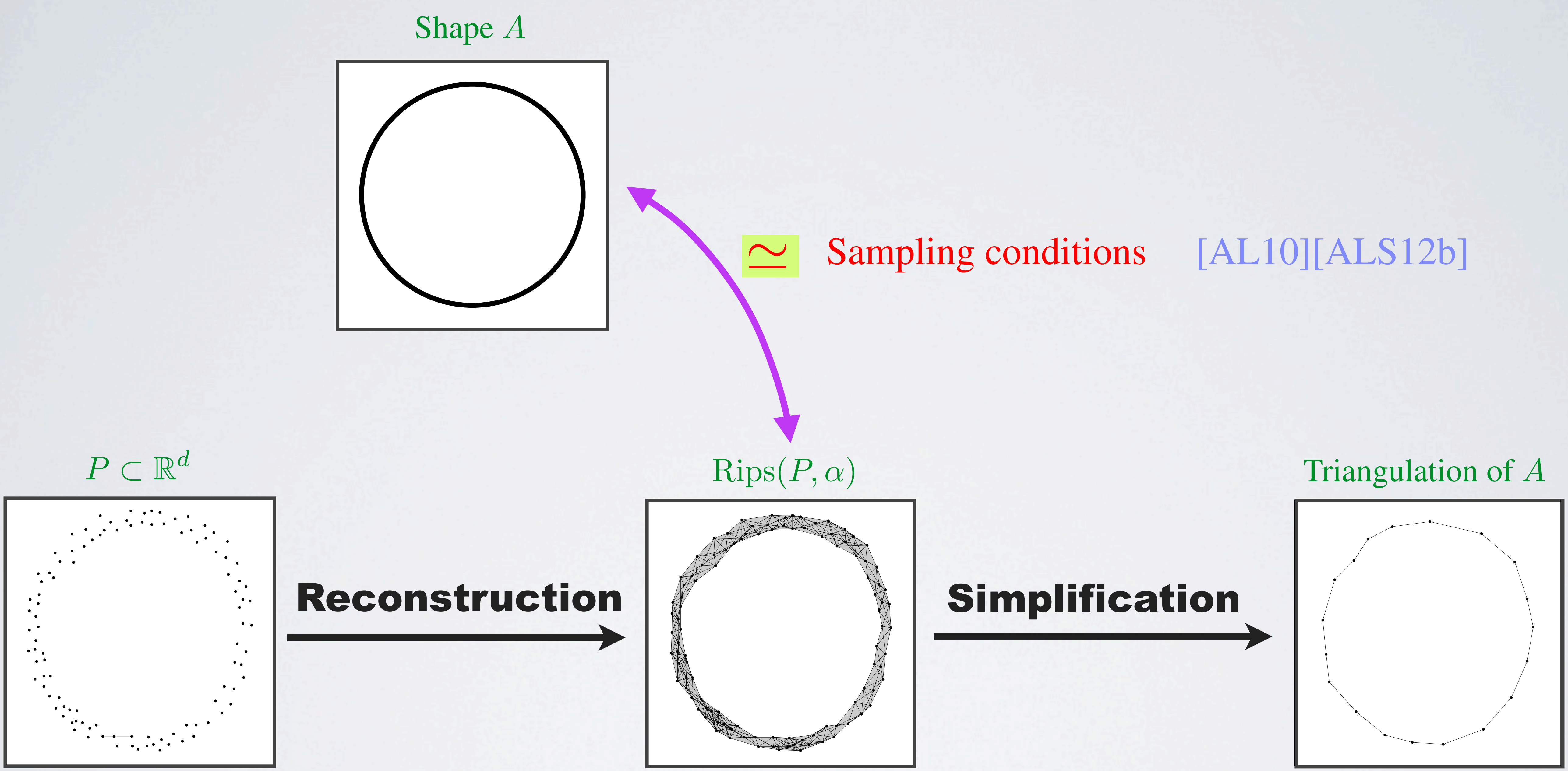
Compressed form of storage
through the 1-skeleton
which is easy to compute

SHAPE RECONSTRUCTION



Compressed form of storage through the 1-skeleton which is easy to compute

SHAPE RECONSTRUCTION



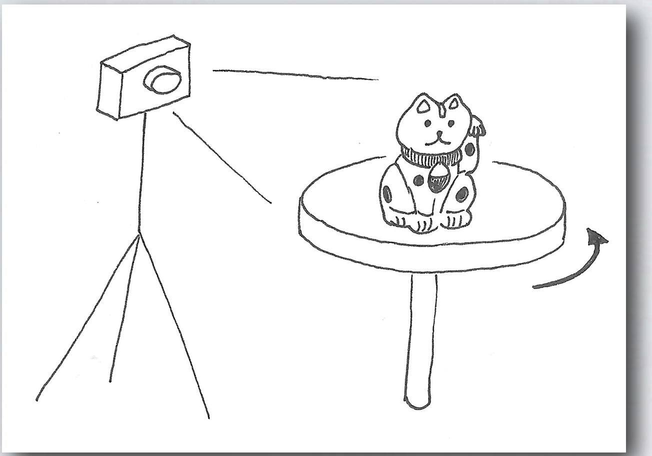
Compressed form of storage through the 1-skeleton which is easy to compute

Can be high-dimensional!

1. **Reduce the size**
2. **Retrieve topology**

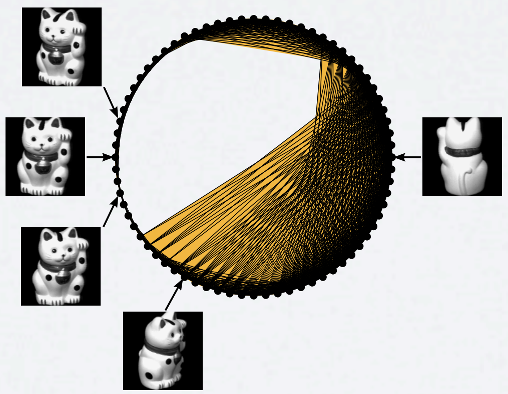
Example

Physical system

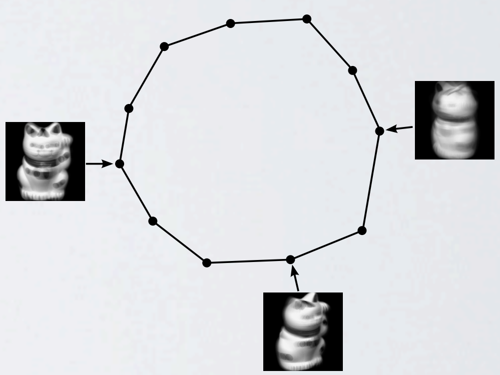


Point cloud in $\mathbb{R}^{128^2}$

Rips complex



Polygonal curve



Correct homotopy type

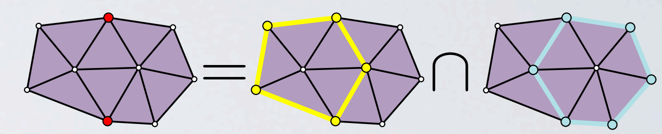
Is high-dimensional!

Correct intrinsic dimension

Simplification by iteratively applying elementary operations

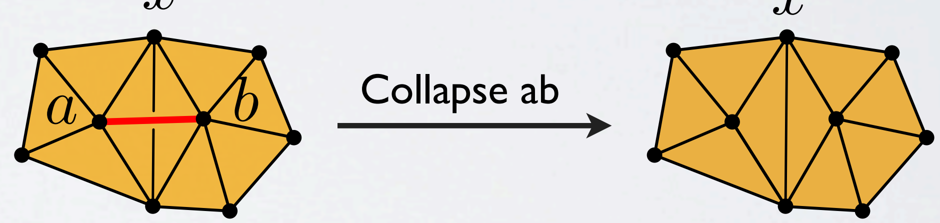
- Edge contraction $ab \mapsto c$

  - Identifies vertices a and b to vertex c
  - Preserves homotopy type if $\mathrm{Lk}_K(ab) = \mathrm{Lk}_K(a) \cap \mathrm{Lk}_K(b)$



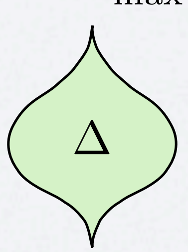
$\mathrm{Lk}_K(\sigma) = \{\tau \mid \tau \cap \sigma = \emptyset, \tau \cup \sigma \in K\}$

- Collapse of a simplex σ_{min}



  - Removes σ_{min} and its cofaces Δ
  - Preserves homotopy type if Δ has a unique maximal element $\sigma_{\mathrm{max}} \neq \sigma_{\mathrm{min}}$

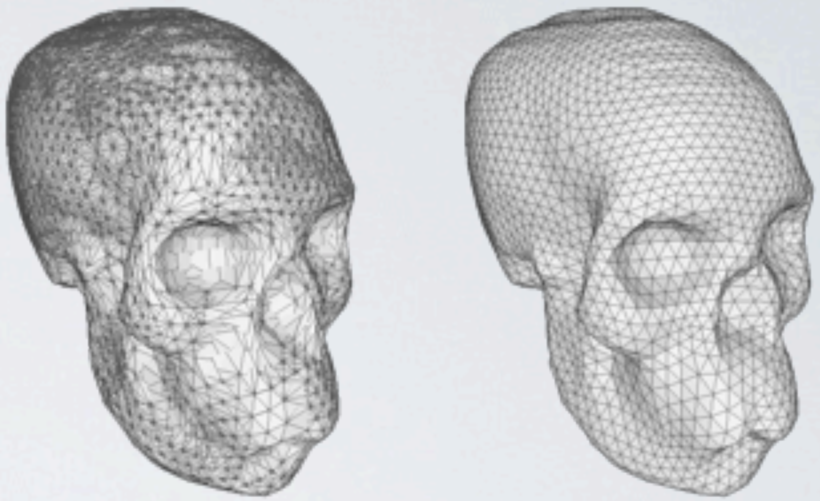
$\sigma_{\mathrm{max}} \neq \sigma_{\mathrm{min}}$



σ_{min}

Does a simplification exist?

- Different strategies:
  - Edge contractions;
  - Vertex and edge collapses;
  - Seems to work well in practice ...



- And yet, not at all obvious that the Rips complex whose vertices sample a shape contains a subcomplex homeomorphic to that shape.
  - A triangulated Bing's house is contractible but not collapsible

  - Geometry has to play a key role.

Simplifying Rips complexes

$A \subset \mathbb{R}^d$ is a compact set

$P \subset \mathbb{R}^d$ is a finite point set

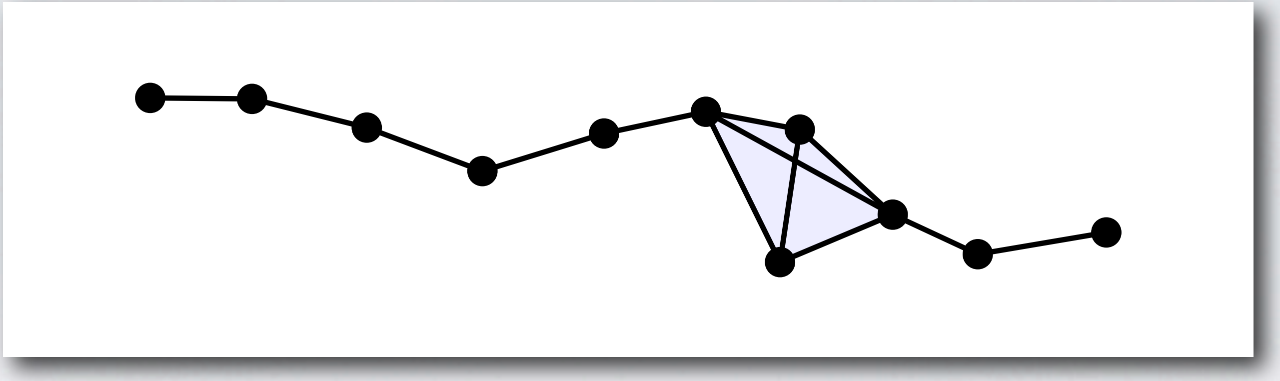
$\alpha > 0$

$$\mathrm{Rips}(P, \alpha) \xrightarrow[\text{Conditions}]{\text{sequence of collapses}} \begin{array}{c}\alpha\text{-nice}\\ \text{triangulation}\\ \text{of shape } A\end{array}$$

- Unfortunately:
  - the proof is not constructive (no algorithm!);
  - it only works for shapes that have an α-nice triangulation!

A key tool

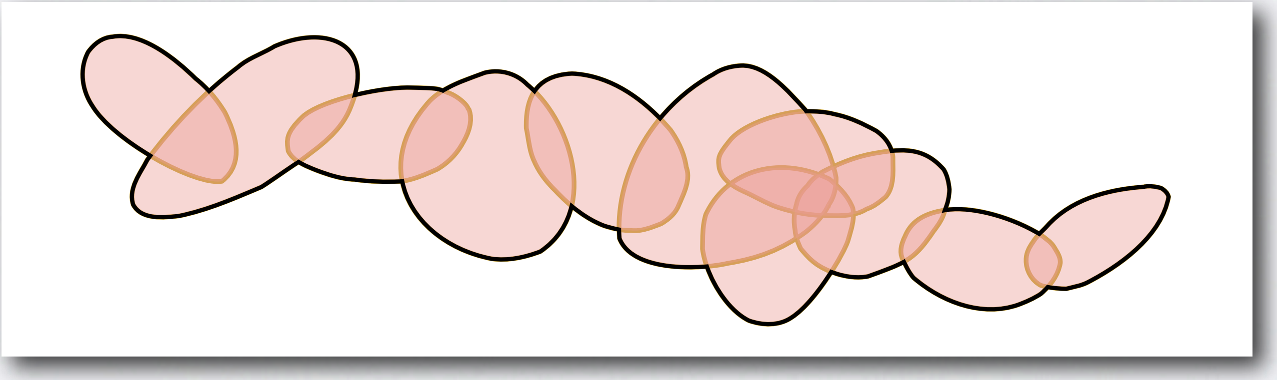
$$\mathrm{Nerve}\,\mathcal{C} = \{\sigma \subset P \mid \sigma \neq \emptyset \text{ and } \bigcap_{p\in\sigma} C_p \neq \emptyset\}$$



If $\bigcap_{z\in\sigma} C_z$ is either empty or contractible

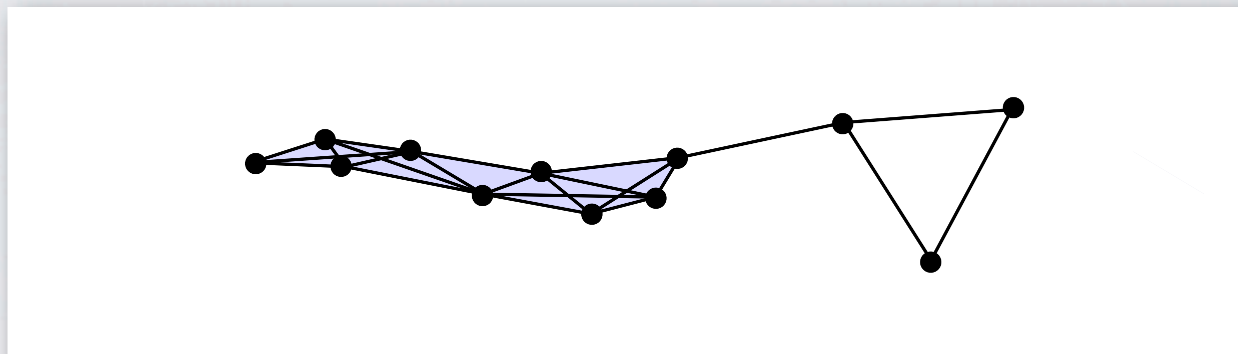
$\simeq$ **Nerve Lemma.**

$\bigcup \mathcal{C}$, where $\mathcal{C} = \{C_p \mid p \in P\}$ finite collection of closed sets



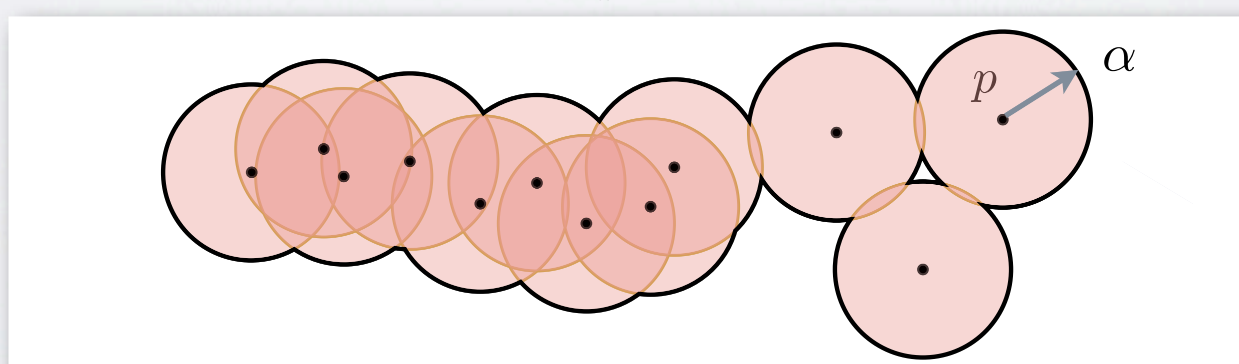
Čech complexes

$$\mathrm{Cech}(P,\alpha) = \mathrm{Nerve}\{B(p,\alpha) \mid p \in P\}$$



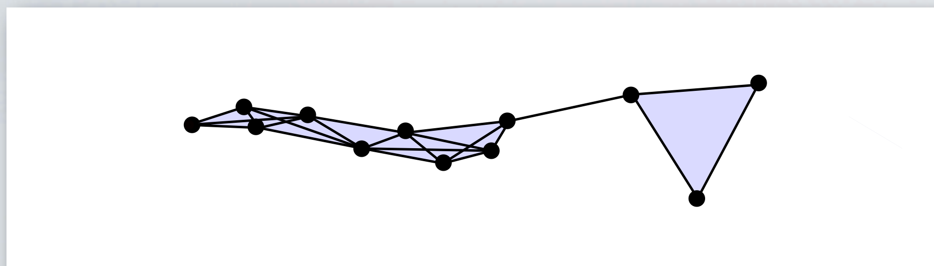
$\simeq$ **Nerve Lemma.**

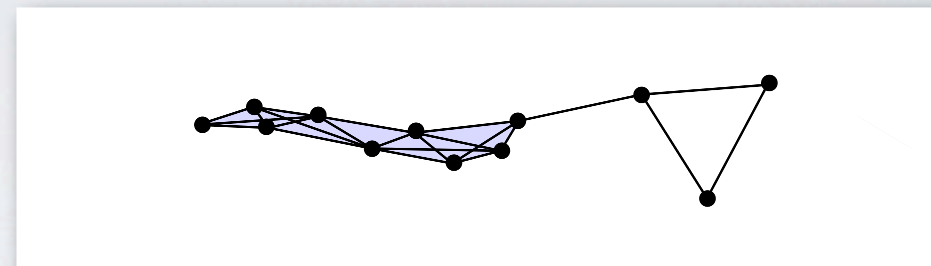
$$P^{\oplus\alpha} = \bigcup_{p\in P} B(p,\alpha) \qquad \alpha\text{-offset of } P$$



Čech complexes

$\mathrm{Rips}(P, \alpha) \quad \supset \quad \mathrm{Cech}(P, \alpha) = \mathrm{Nerve}\{B(p, \alpha) \mid p \in P\}$

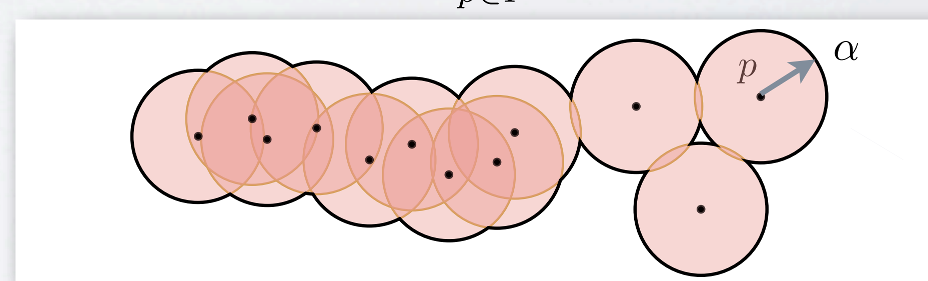




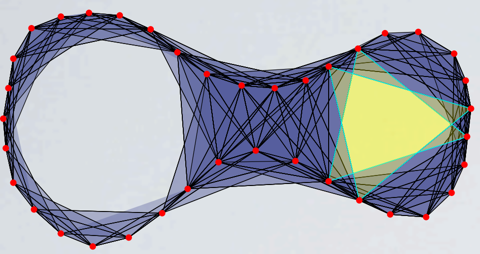
$\simeq$ **Nerve Lemma.**

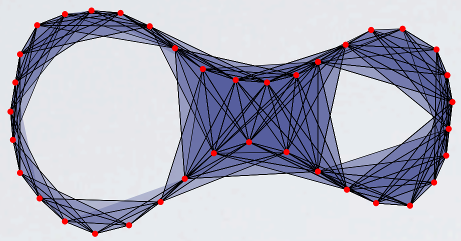
$$P^{\oplus\alpha} = \bigcup_{p \in P} B(p, \alpha)$$

α-offset of P



Overview of what we knew!

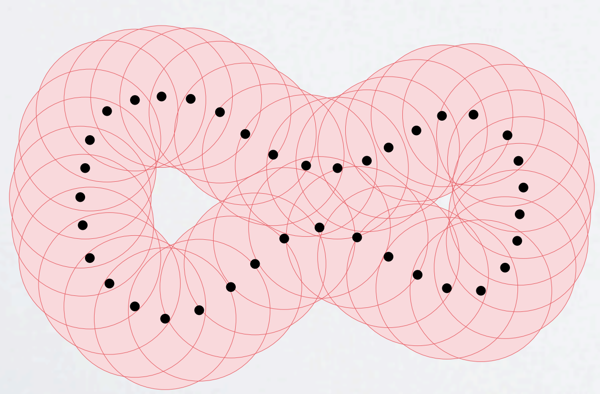




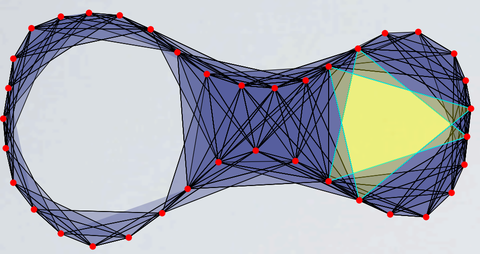
$\mathrm{Rips}(P, \alpha)$ $\supset$ $\mathrm{Cech}(P, \alpha)$

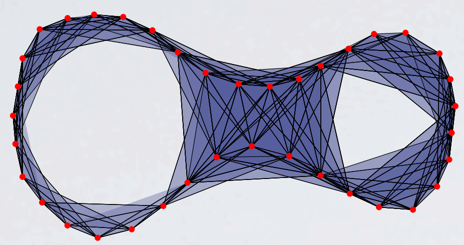
$\simeq$ Nerve Lemma

$P^{\oplus \alpha}$



Overview of what we knew!





$\text{Rips}(P,\alpha) \supset \text{Cech}(P,\alpha)$

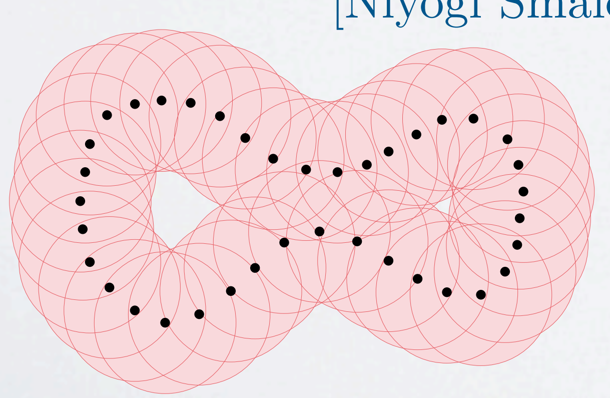
$\simeq$ Nerve Lemma (between $\text{Cech}(P,\alpha)$ and $P^{\oplus\alpha}$)

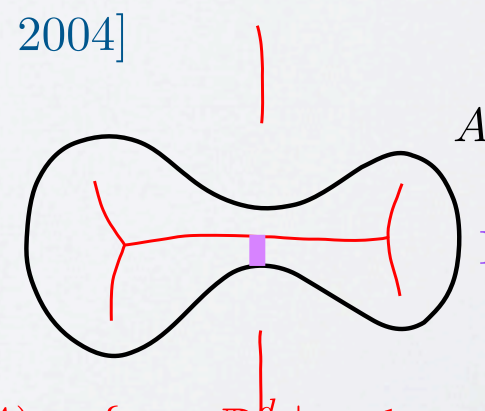
$P^{\oplus\alpha} \xrightarrow{\text{deform. retracts}} \text{Shape } A$

(SC1) $d_H(A,P) \leq \varepsilon < (3-\sqrt{8})\,\text{Reach}\,A$

$\alpha = (2+\sqrt{2})\varepsilon$

[Niyogi Smale Weinberger 2004]

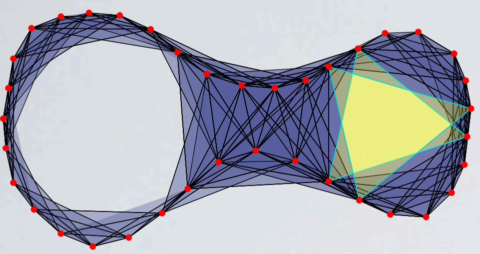


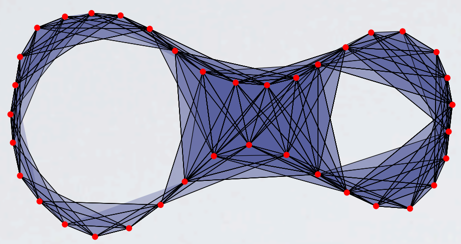


$\text{Reach}\,A = d(A, \text{MedialAxis}(A))$

$\text{MedialAxis}(A) = \{ m \in \mathbb{R}^d \mid m \text{ has at least two closest points in } A \}$

Overview of what we knew!





$\text{Rips}(P, \alpha) \xrightarrow{\text{collapses}} \text{Cech}(P, \alpha)$

(SC2) $d_H(A, P) \leq \varepsilon < \frac{2\sqrt{2-\sqrt{2}} - \sqrt{2}}{2+\sqrt{2}} \text{Reach}(A)$

$\alpha \approx 7.22\varepsilon$

[ALS12b]

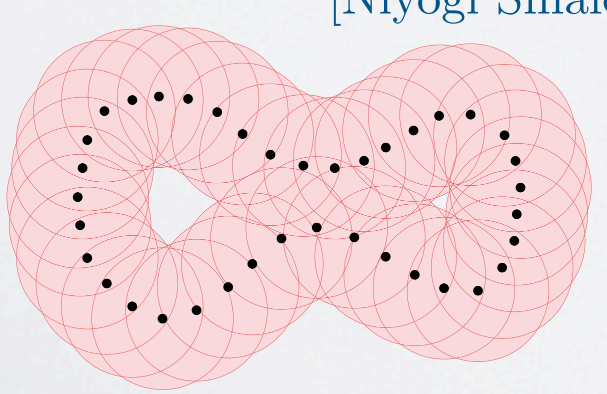
$\text{Cech}(P, \alpha) \simeq P^{\oplus\alpha}$ Nerve Lemma

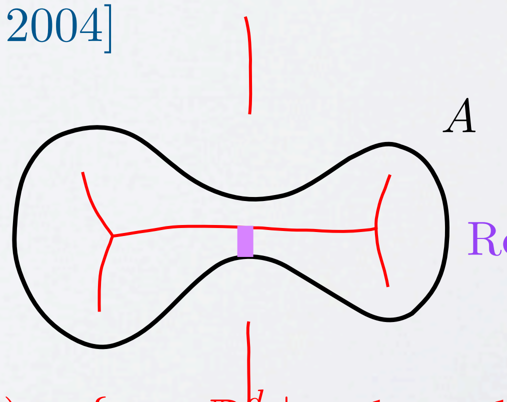
$P^{\oplus\alpha} \xrightarrow{\text{deform. retracts}} \text{Shape } A$

(SC1) $d_H(A, P) \leq \varepsilon < (3 - \sqrt{8}) \, \text{Reach } A$

$\alpha = (2 + \sqrt{2})\varepsilon$

[Niyogi Smale Weinberger 2004]





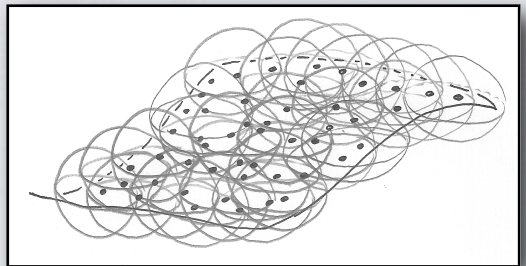
$\text{Reach } A = d(A, \text{MedialAxis}(A))$

$\text{MedialAxis}(A) = \{ m \in \mathbb{R}^d \mid m \text{ has at least two closest points in } A \}$

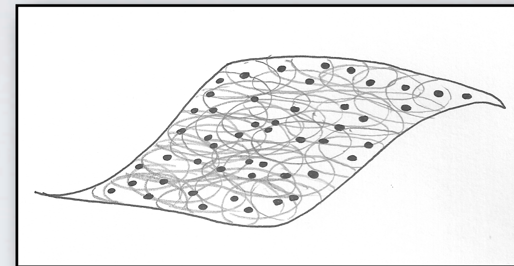
Overview of what we knew!

$$
\begin{tikzcd}
\mathrm{Rips}(P,\alpha) \arrow[r, "\text{collapses}", "\substack{\text{(SC2)}\\ \text{[ALS12b]}}"'] & \mathrm{Cech}(P,\alpha) \arrow[d, leftrightarrow, "\simeq \text{ Nerve Lemma}"] & \\
& P^{\oplus\alpha} \arrow[r, "\text{deform. retracts}", "\substack{\text{(SC1)}\\ \text{[NSW04]}}"'] & \text{Shape } A
\end{tikzcd}
$$

Overview of what is new



Nerve$\{B(p,\alpha) \mid p \in P\}$



Nerve$\{A \cap B(p,\alpha) \mid p \in P\}$

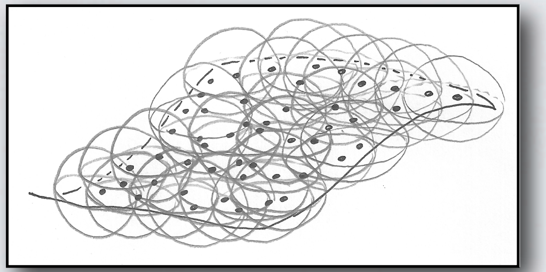
$\mathrm{Rips}(P,\alpha) \xrightarrow{\text{collapses}} \mathrm{Cech}(P,\alpha)$ (SC2) [ALS12b]

$\mathrm{Cech}_A(P,\alpha)$

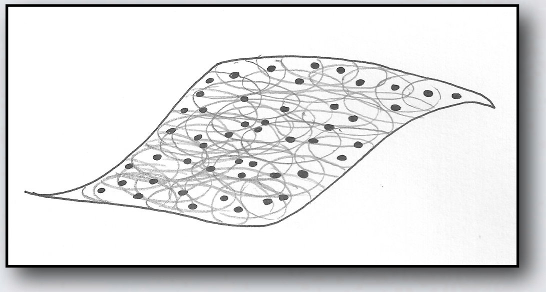
$\mathrm{Cech}(P,\alpha) \simeq P^{\oplus\alpha}$ Nerve Lemma

$P^{\oplus\alpha} \xrightarrow{\text{deform. retracts}} \text{Shape } A$ (SC1) [NSW04]

Overview of what is new



$\mathrm{Nerve}\{B(p,\alpha) \mid p \in P\}$



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\end{tikzcd}$$

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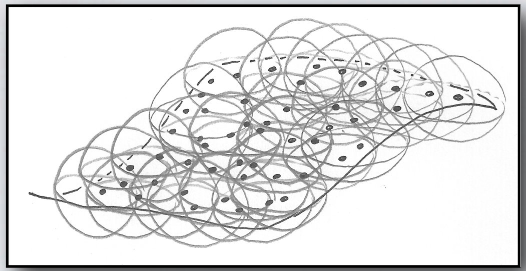

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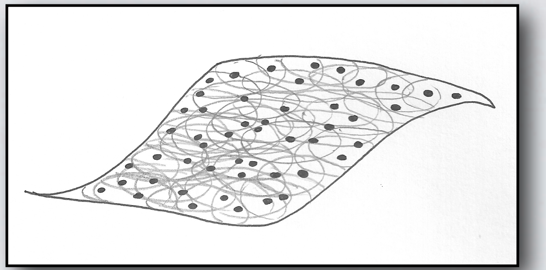

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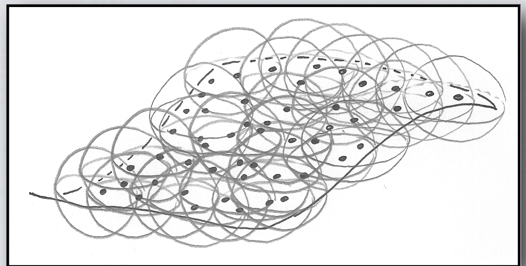

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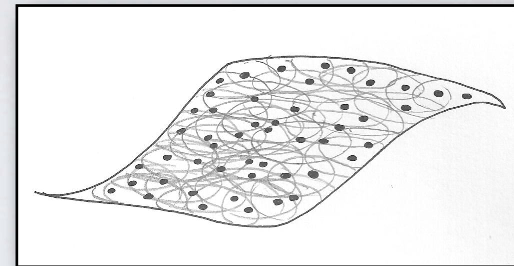

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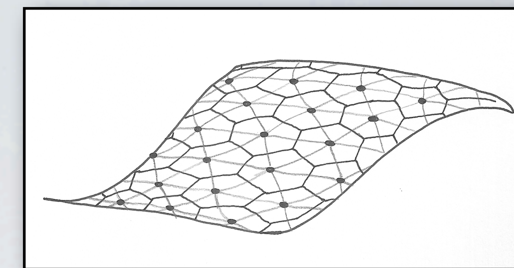

$\mathrm{Nerve}\{A \cap \mathrm{Hull}_\alpha(C_v) \mid v \in V\}$

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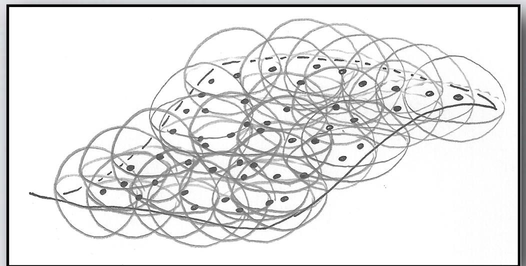

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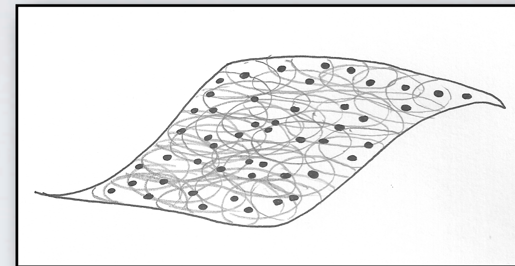

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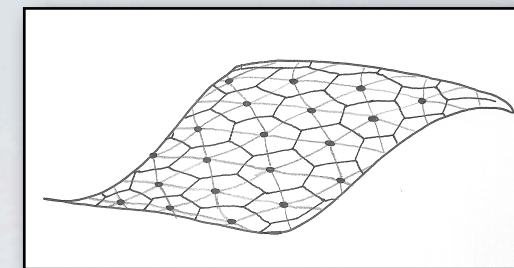

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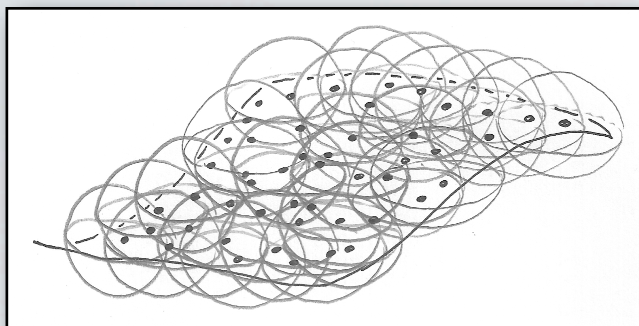
\end{tikzcd}
$$

Restricting the Čech complex

Theorem 2 *If* $d_H(A, P) \leq \varepsilon < (3 - \sqrt{8})\,\mathrm{Reach}(A)$ *and* $\alpha = (2 + \sqrt{2})\varepsilon$, *then there exists a sequence of collapses from* $\mathrm{Cech}(P, \alpha)$ *to* $\mathrm{Cech}_A(P, \alpha)$.

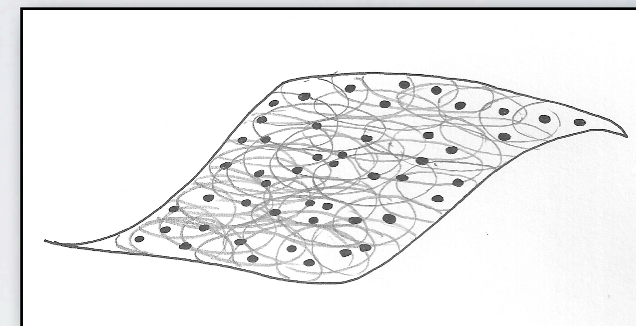


$\mathrm{Nerve}\{B(p,\alpha) \mid p \in P\}$

$\|$

$K(+\infty)$

$$\mathrm{Cech}(P,\alpha) \xrightarrow{\text{Define collapses?}} \mathrm{Cech}_A(P,\alpha)$$

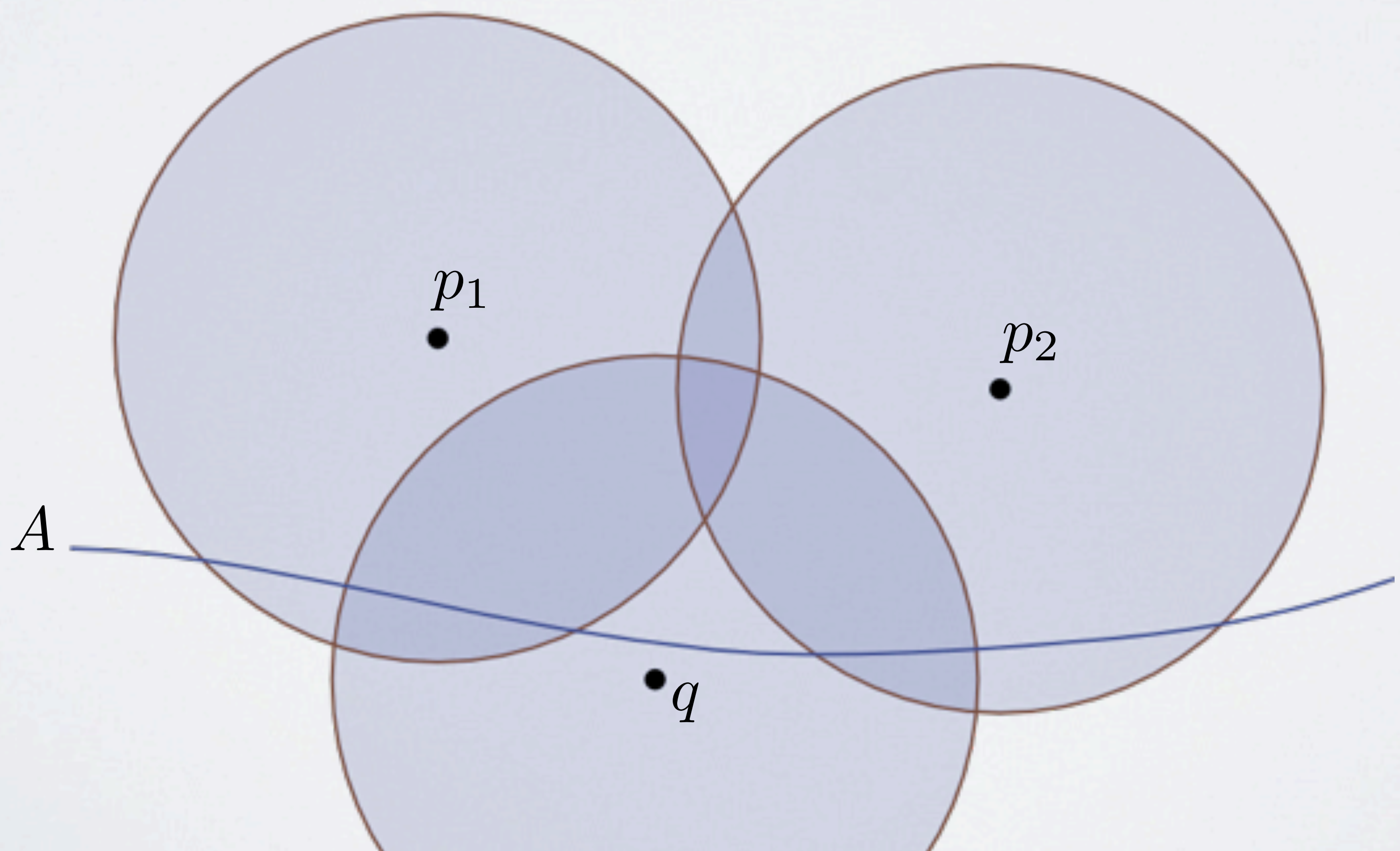


$\mathrm{Nerve}\{A \cap B(p,\alpha) \mid p \in P\}$

$\|$

$K(0)$

$$K(t) = \mathrm{Nerve}\{A^t \cap B(p,\alpha) \mid p \in P\}$$



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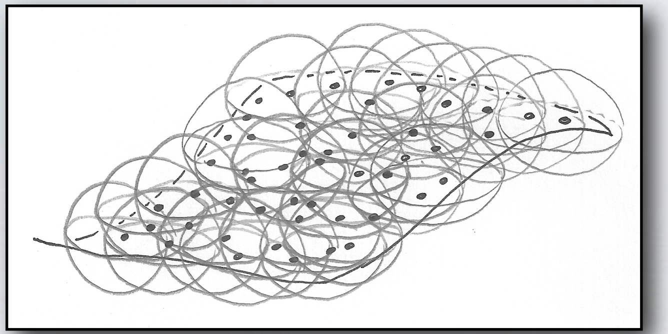
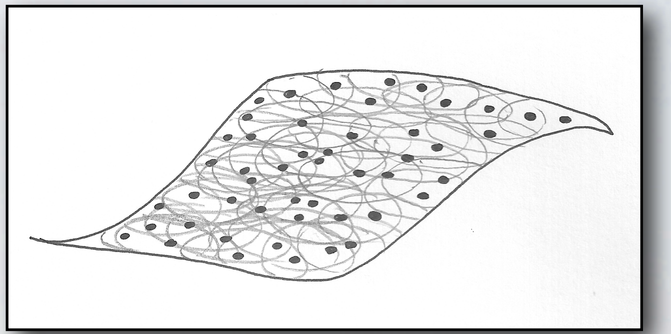


$\mathrm{Cech}(P, \alpha)$ $\xrightarrow{\text{Define collapses?}}$ $\mathrm{Cech}_A(P, \alpha)$



$\mathrm{Nerve}\{B(p, \alpha) \mid p \in P\} = K(+\infty)$

$\mathrm{Nerve}\{A \cap B(p, \alpha) \mid p \in P\} = K(0)$

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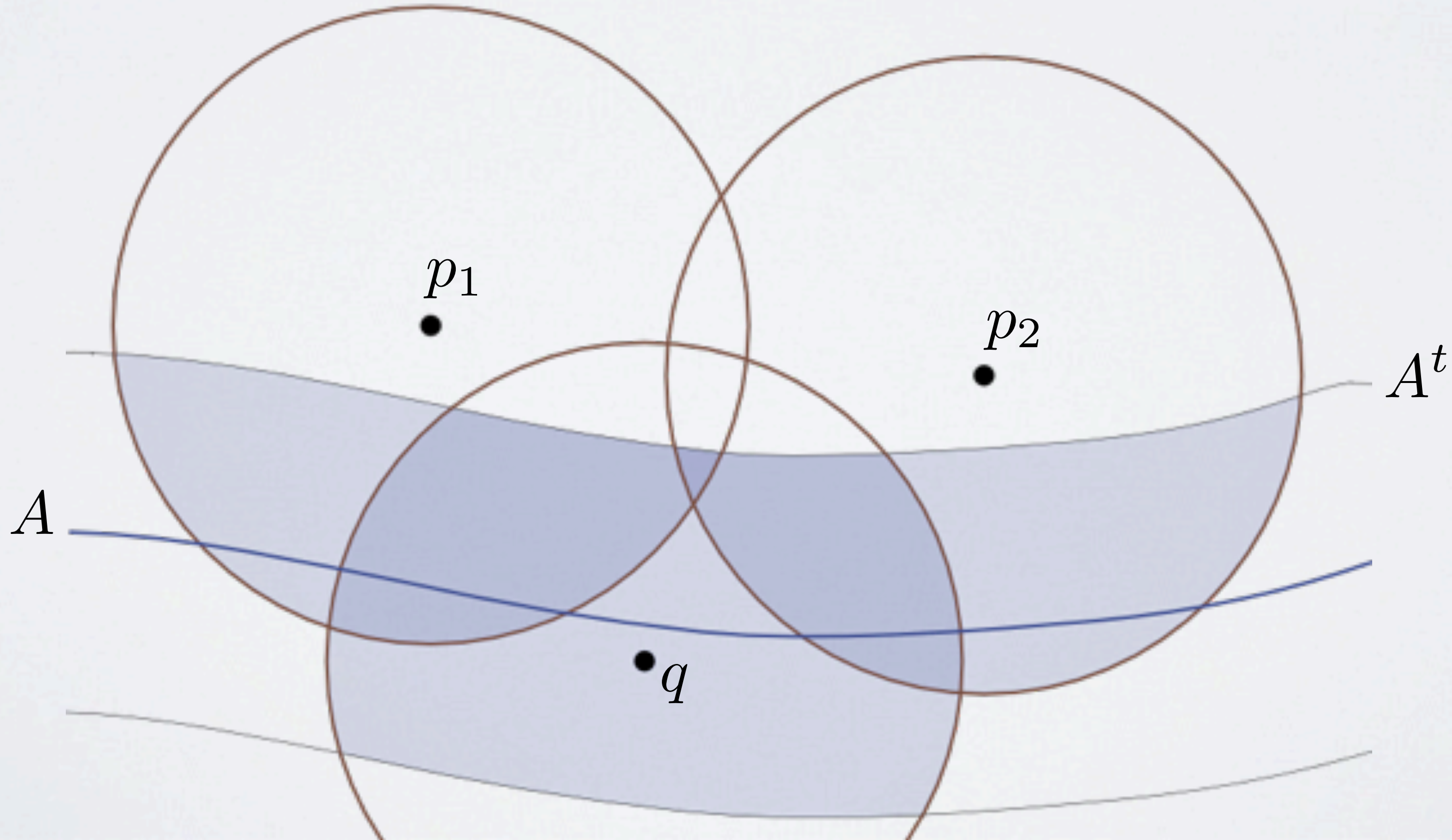

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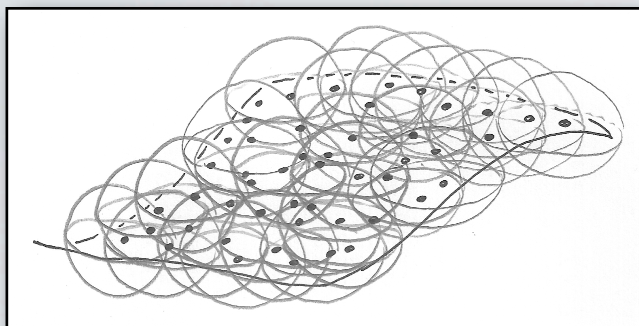

$\mathrm{Nerve}\{B(p, \alpha) \mid p \in P\}$

$\|$

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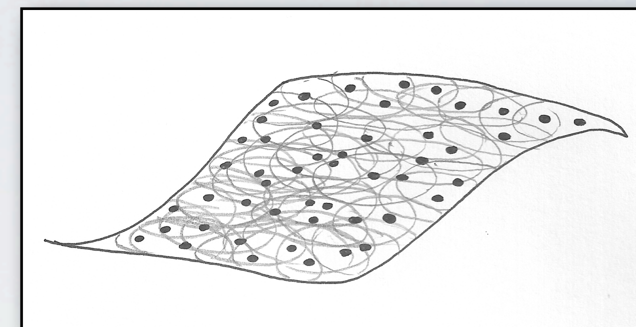

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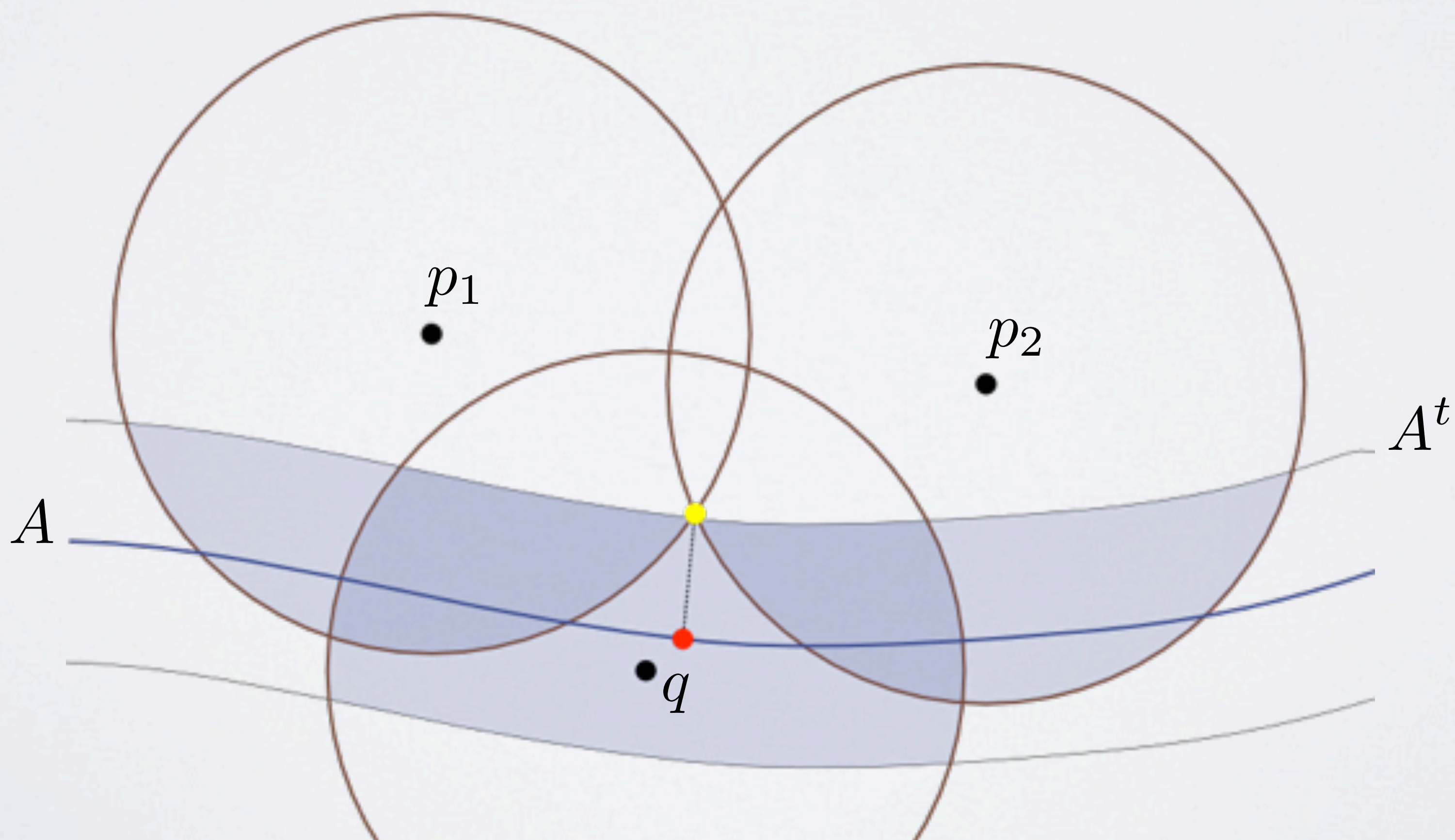

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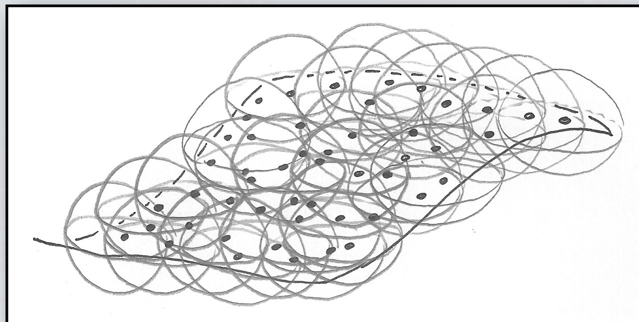

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$\|$

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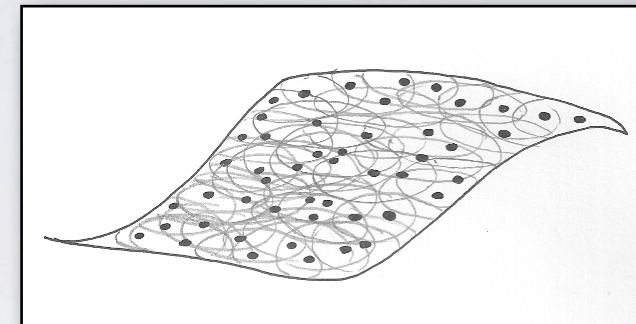

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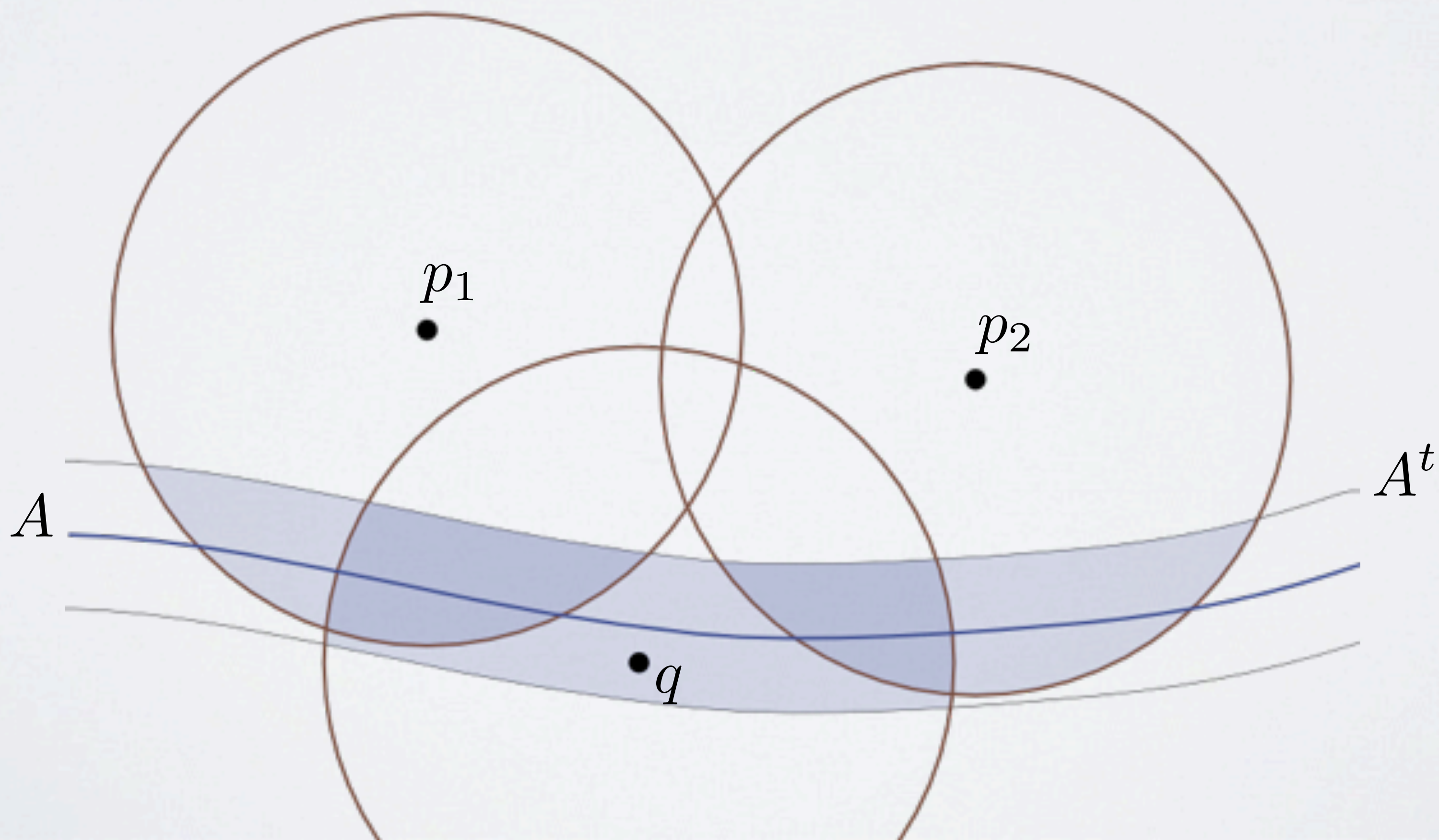


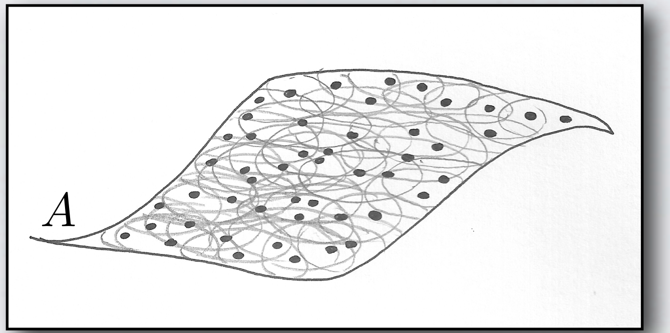
The restricted Čech complex

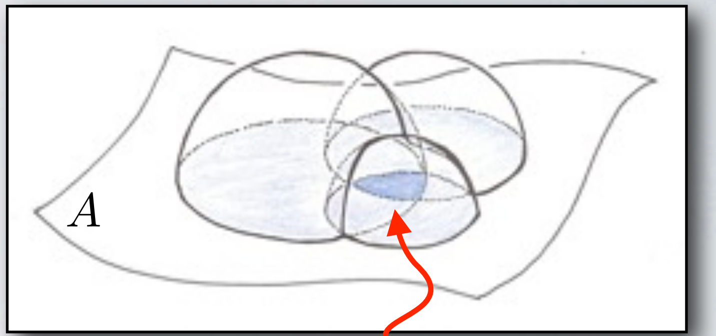
Theorem 1 *If* $\alpha < \mathrm{Reach}(A)$ *and* $A \subset P^{\oplus\alpha}$, *then* $\mathrm{Cech}_A(P,\alpha) \simeq A$.

$\mathrm{Cech}_A(P,\alpha) = \mathrm{Nrv}\{A \cap B(p,\alpha) \mid p \in P\}$

$\simeq ?$ Apply the Nerve Lemma

$A = \bigcup_{p \in P} [A \cap B(p,\alpha)]$





contractible?

Technical Lemma. $A \cap \bigcap_{z \in \text{compact subset } \sigma} B(z,\alpha)$ is either empty or contractible if $\alpha < \mathrm{Reach}(A)$

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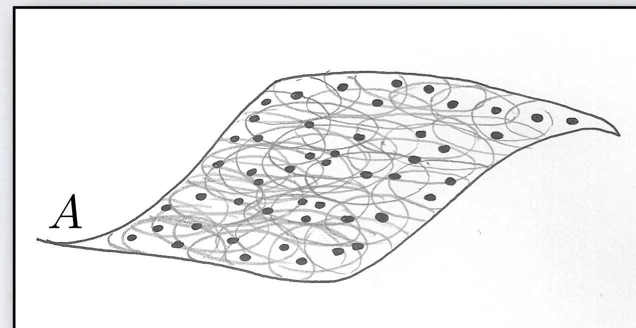

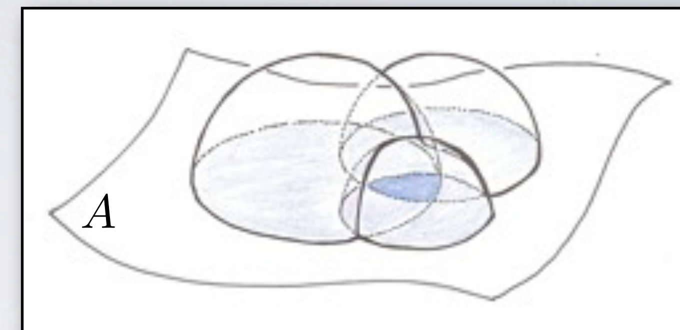

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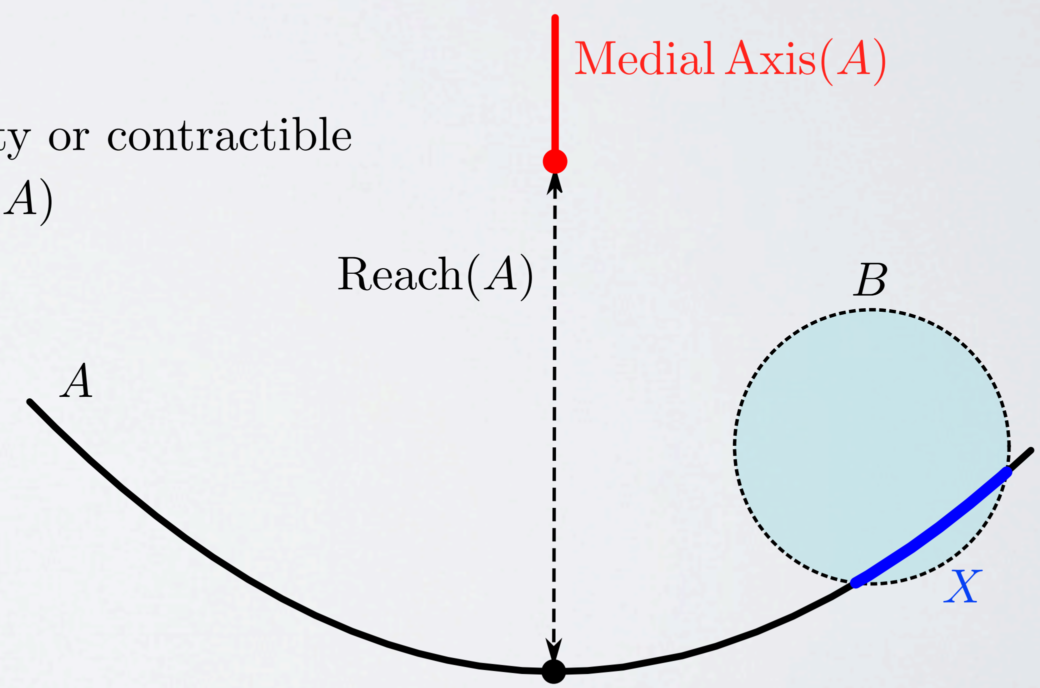
- ⋆ $\mathrm{Reach}(A) \leq \mathrm{Reach}(A \cap B)$ whenever $\mathrm{Rad}(B) \leq \mathrm{Reach}(A)$
- ⋆ if $\mathrm{Rad}(X) \leq \mathrm{Reach}(X)$, then X contractible



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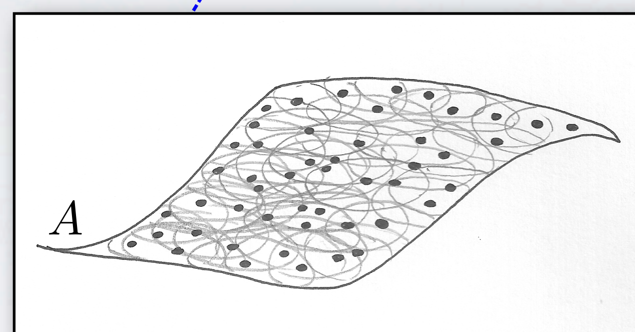

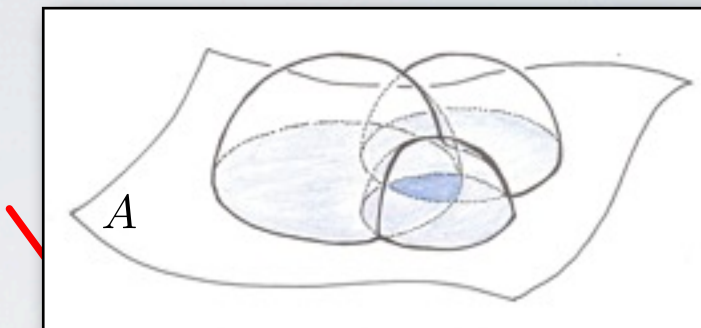

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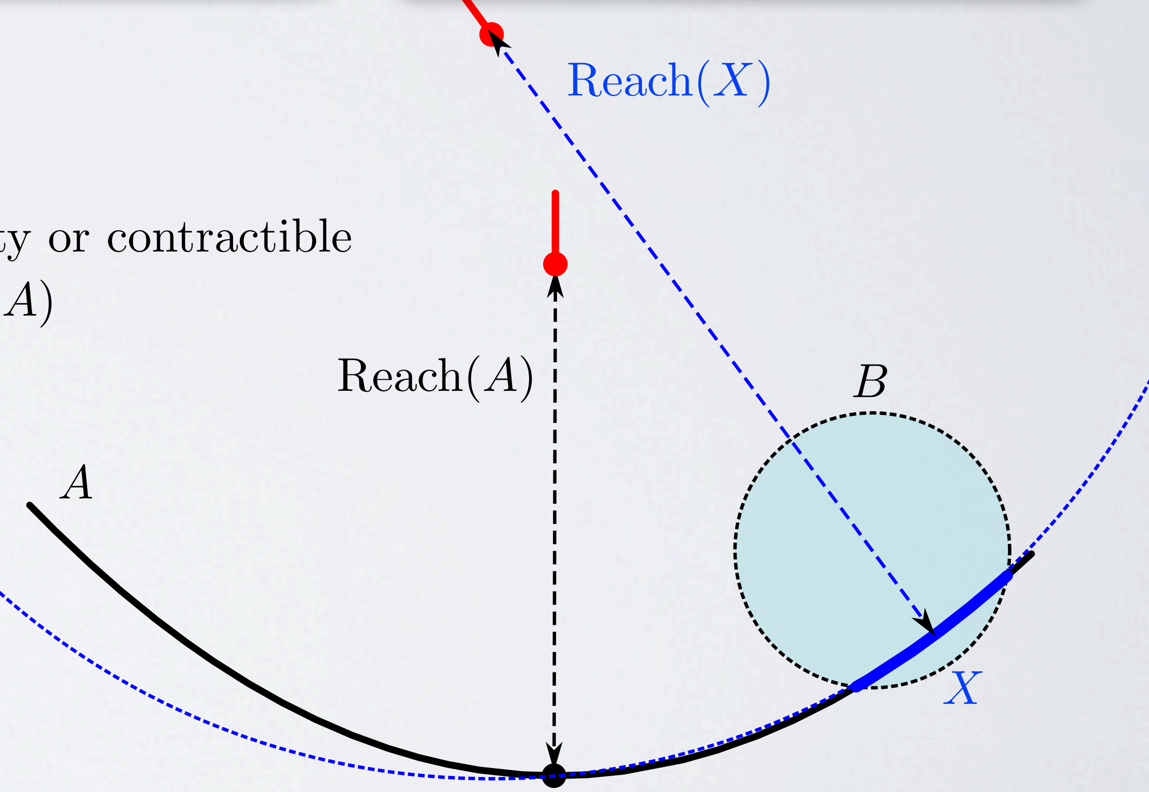

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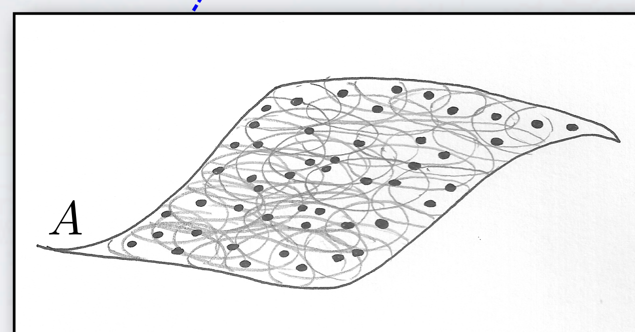

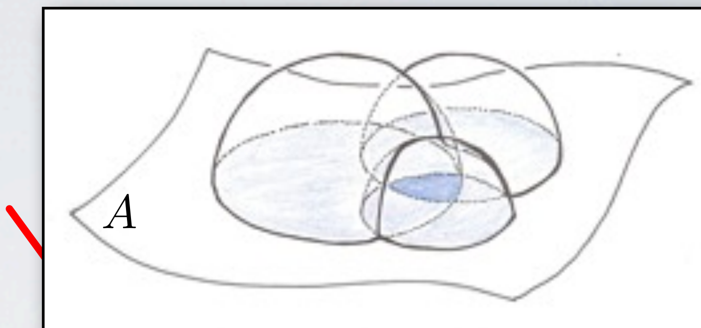

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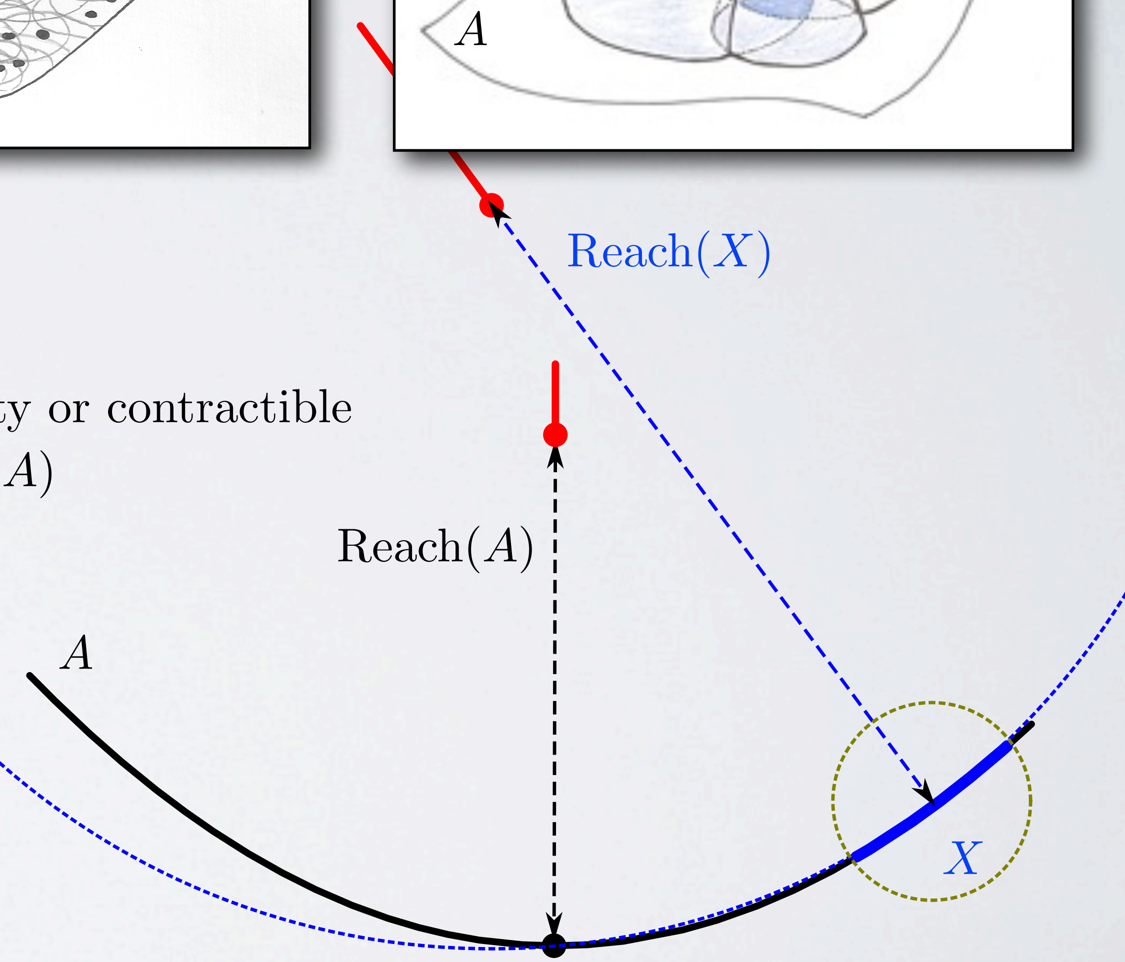

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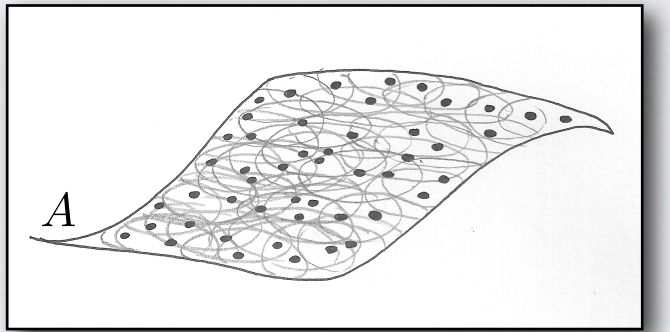

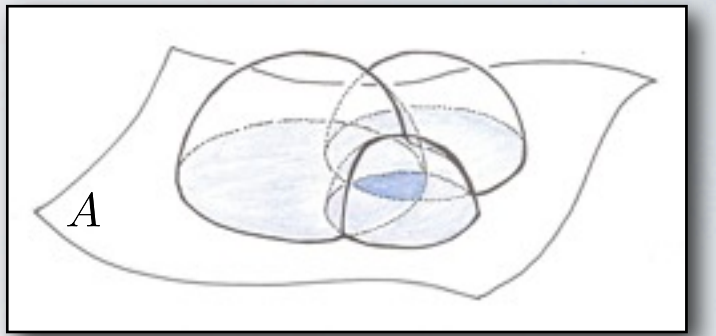

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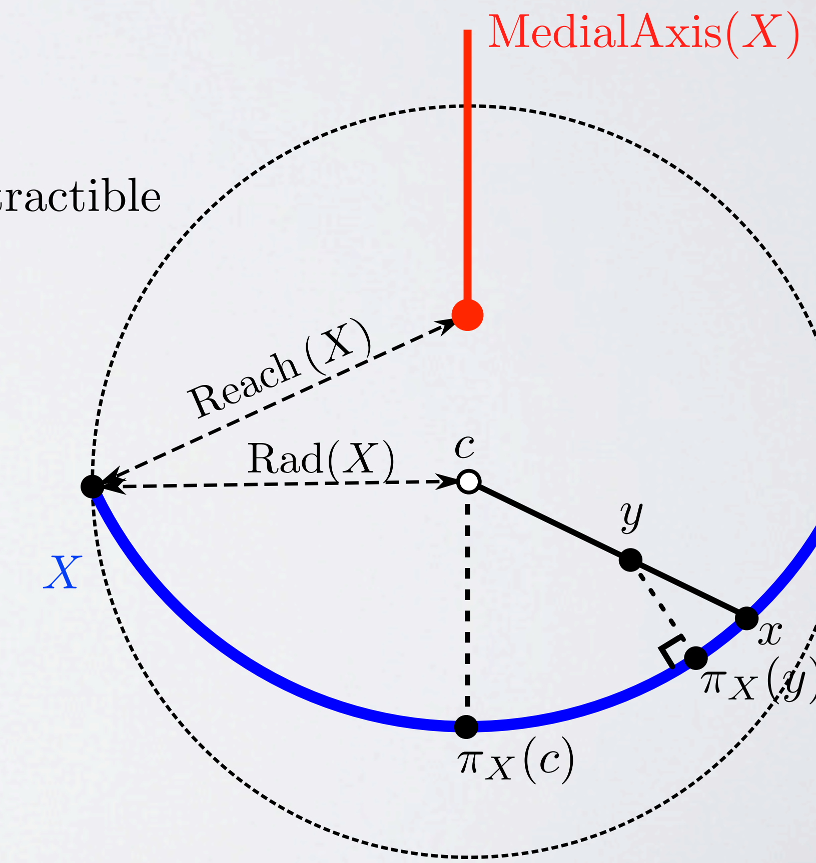

$$H(t,x) = \pi_X((1-t)x + tc)$$

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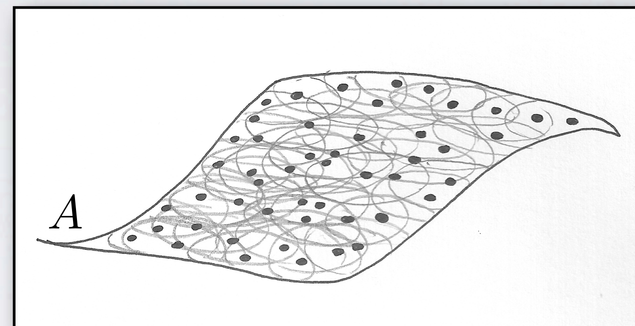
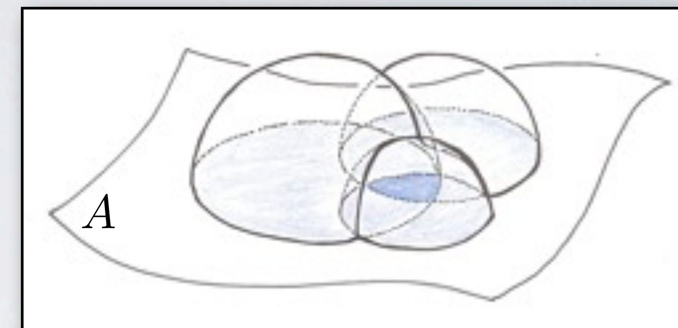
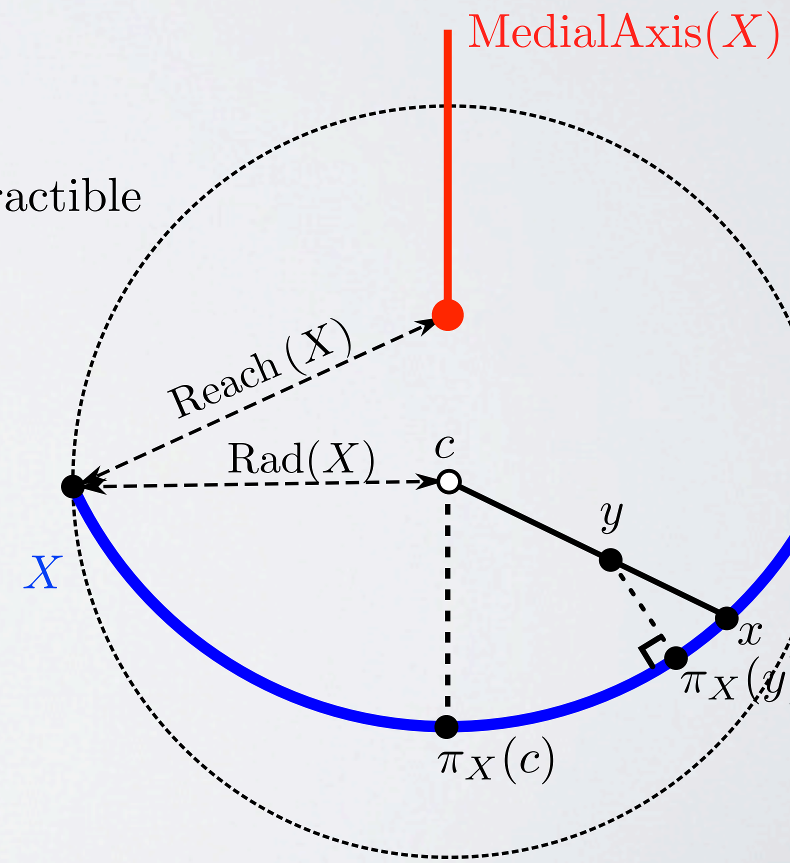




Technical Lemma. $A \cap \bigcap_{z \in \text{compact subset } \sigma} B(z,\alpha)$ is either empty or contractible if $\alpha < \mathrm{Reach}(A)$



$H(t,x) = \pi_X((1-t)x + tc)$

- $\star$ $\mathrm{Reach}(A) \leq \mathrm{Reach}(A \cap B)$ whenever $\mathrm{Rad}(B) \leq \mathrm{Reach}(A)$
- $\star$ if $\mathrm{Rad}(X) \leq \mathrm{Reach}(X)$, then X contractible

$$\mathrm{Rad}(A \cap \bigcap_{z \in \sigma} B(z,\alpha)) \leq \alpha < \mathrm{Reach}(A) \leq \mathrm{Reach}(A \cap \bigcap_{z \in \sigma} B(z,\alpha))$$

α-robust coverings

Technical Lemma. $A \cap \bigcap_{z \in \text{compact subset } \sigma} B(z, \alpha)$ is either empty or contractible if $\alpha < \mathrm{Reach}(A)$

$X \subset A$ with $\mathrm{Rad}(X) < \alpha$ $\qquad\qquad$ $A \cap \mathrm{Hull}_\alpha(X)$

not necessarily contractible! $\qquad\qquad$ contractible if $\alpha < \mathrm{Reach}(A)$

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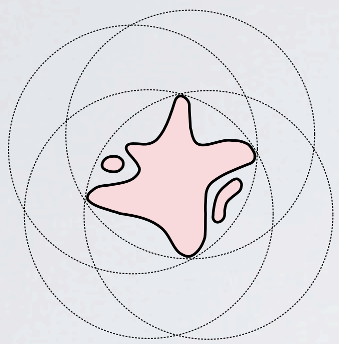

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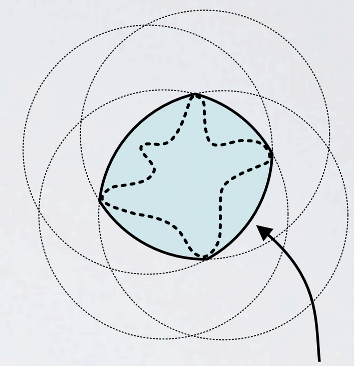
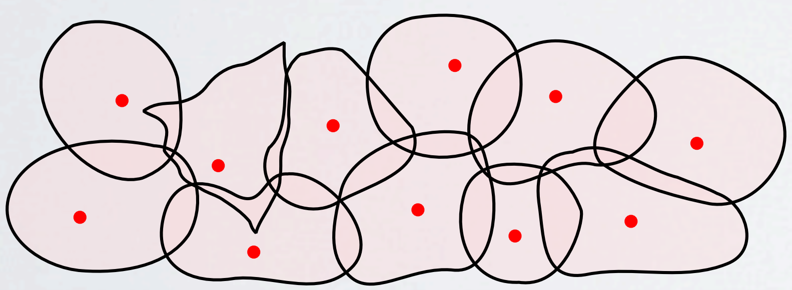
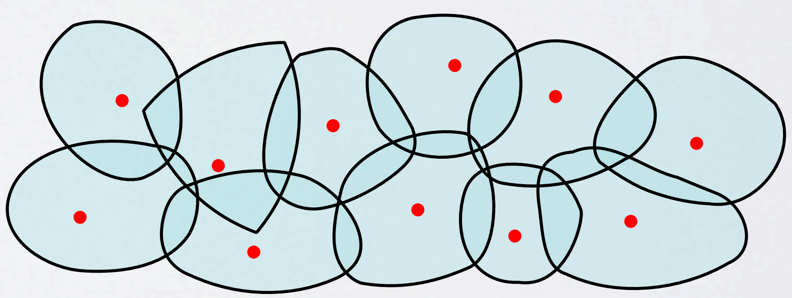


contractible
if $\alpha < \mathrm{Reach}(A)$

$\mathcal{C} = \{C_v \mid v \in V\}$



$\mathcal{C}' = \{A \cap \mathrm{Hull}_\alpha(C_v) \mid v \in V\}$

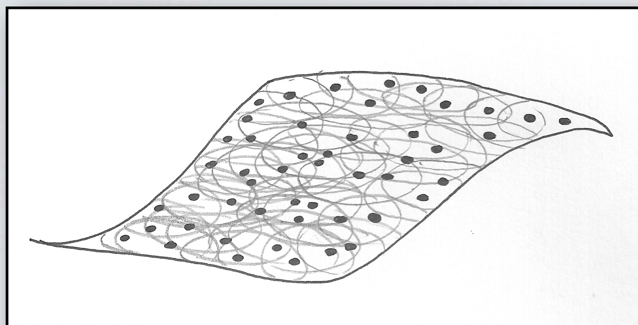


$\mathcal{C}$ α-robust covering of A
if $\mathrm{Nrv}\,\mathcal{C} = \mathrm{Nrv}\,\mathcal{C}'$

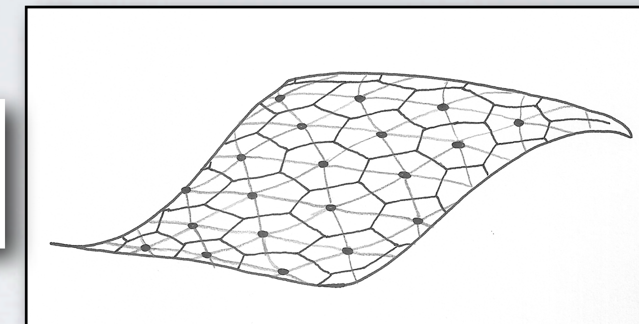
$\mathrm{Nrv}\,\mathcal{C}' \simeq A$
if $\alpha < \mathrm{Reach}(A)$

Collapsing restricted Čech complex

Theorem 3 *Let $\mathcal{C} = \{C_v \mid v \in V\}$ an α-robust covering of A with $V \subset P$. Suppose there exists $f : V \to P$ injective such that $C_v \subset B^\circ(f(v), \alpha)$. If $\alpha < \mathrm{Reach}(A)$, then there is a sequence of collapses from $\mathrm{Cech}_A(P, \alpha)$ to $\mathrm{Nrv}\,\mathcal{C}$.*



$\mathrm{Cech}_A(P, \alpha)$ $\xrightarrow{\text{Define collapses?}}$ Nerve of an α-robust covering of A



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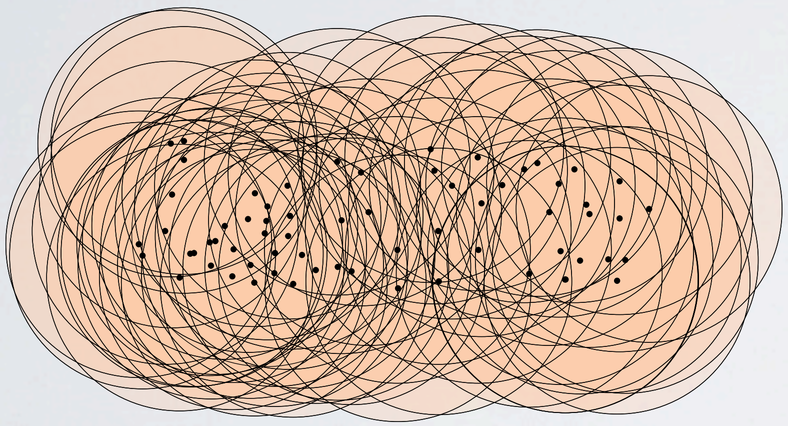
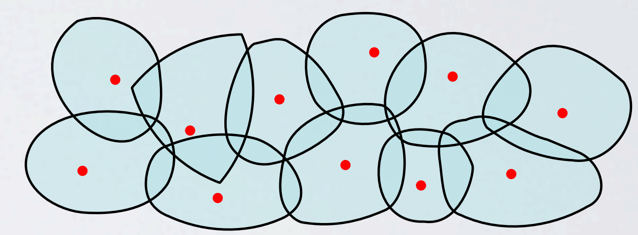
$\mathrm{Cech}_A(P, \alpha)$ $\xrightarrow{\text{Define collapses?}}$ Nerve of an α-robust covering of A

$$K(t) = \mathrm{Nerve}\{D_p(t) \mid p \in P\}$$

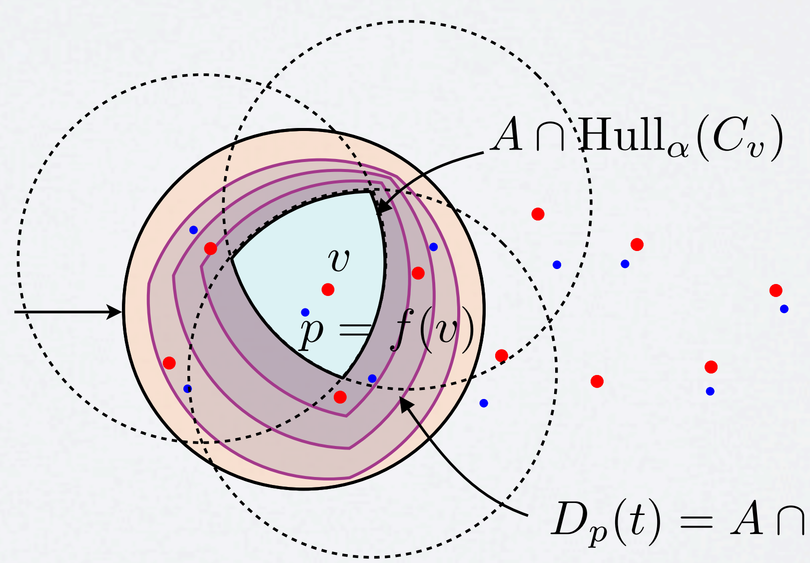


$\mathrm{Nerve}\{A \cap B(p, \alpha) \mid p \in P\}$
$\|$
$K(0)$

$\mathcal{C} = \{C_v \mid v \in V\}$



$\mathrm{Nerve}\{A \cap \mathrm{Hull}_\alpha(C_v) \mid v \in V\}$
$\|$
$K(1)$



$$D_p(t) = A \cap \bigcap_{C_v \subset B(z,\alpha)} B(tz + (1-t)p, \alpha)$$

Evolving family of compact sets

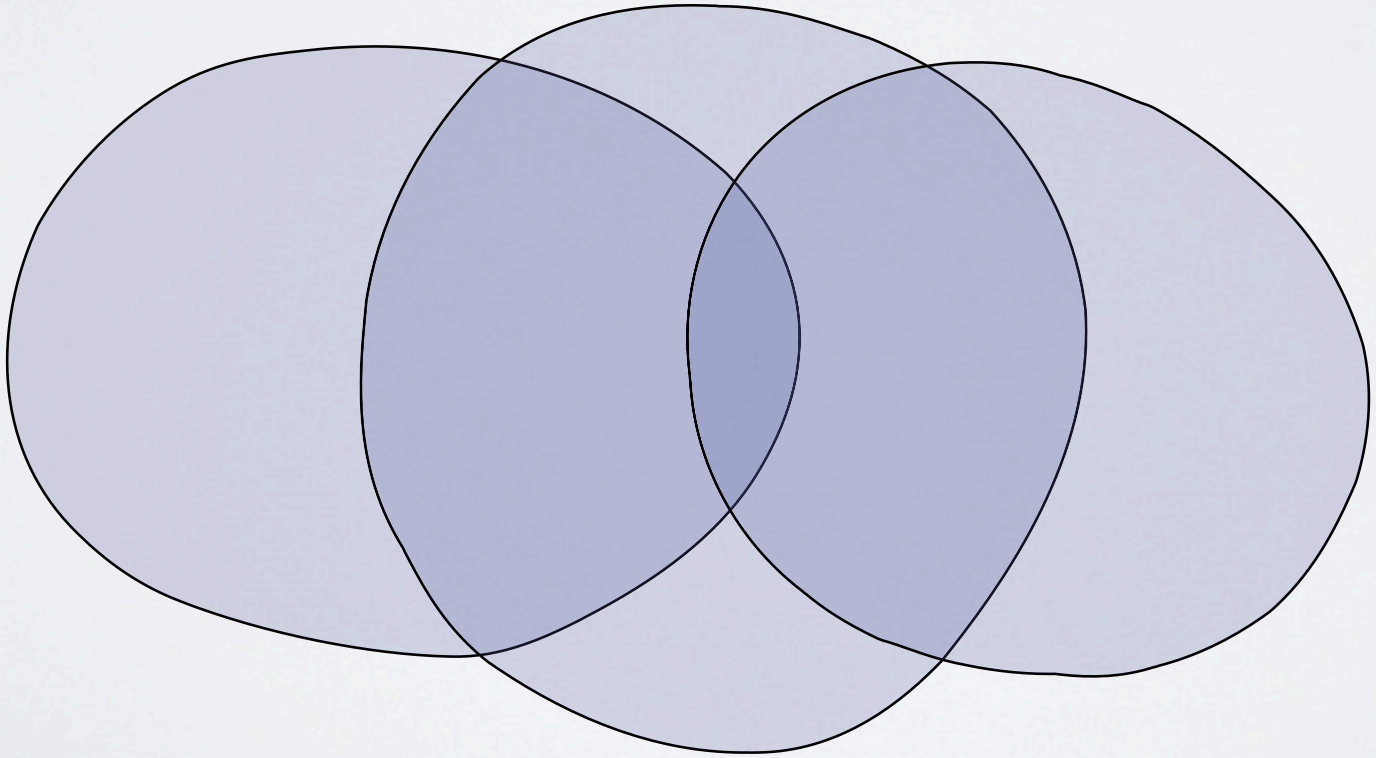
Suppose $K(t) = \text{Nerve}\{D_p(t) \mid p \in P\}$ such that $\forall t_1 < t_2$, $\forall \sigma \subset P$, $\forall t$

(a) $\bigcap_{p\in\sigma} D_p(t)$ empty or connected ;

(b) before disappearing $\bigcap_{p\in\sigma} D_p(t)$ is reduced to a single point.

(c) $D_p(t_2) \subset D_p(t_1)^\circ$;

Then, generically $K(t)$ undergoes collapses as t increases.



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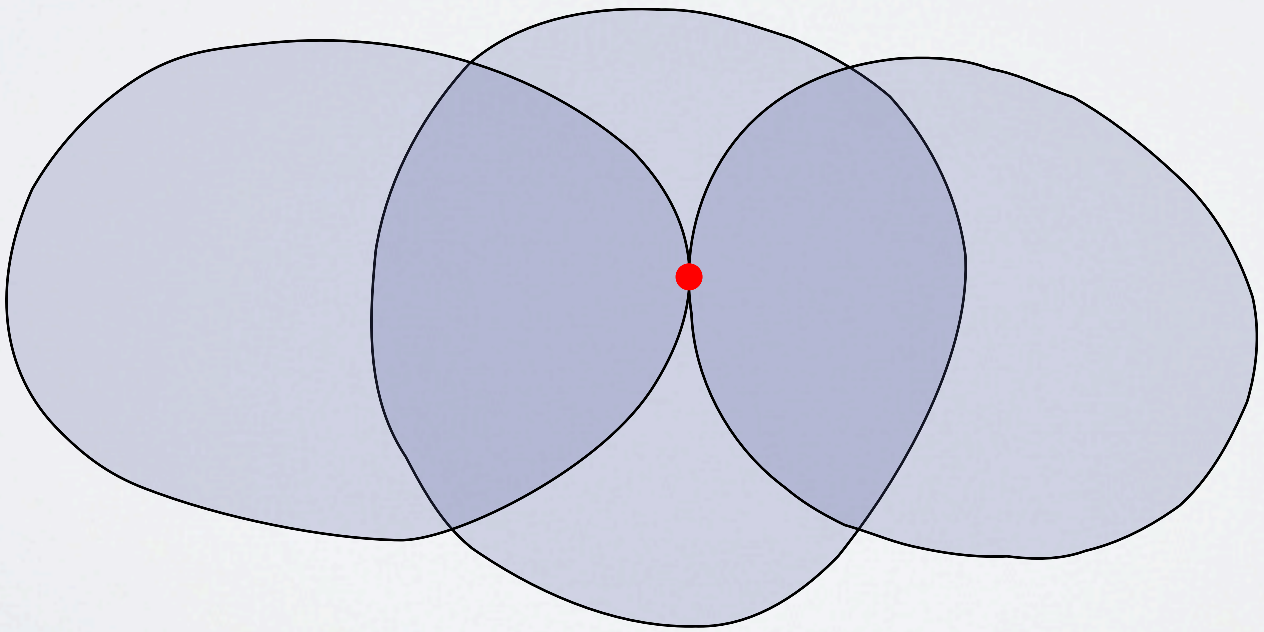

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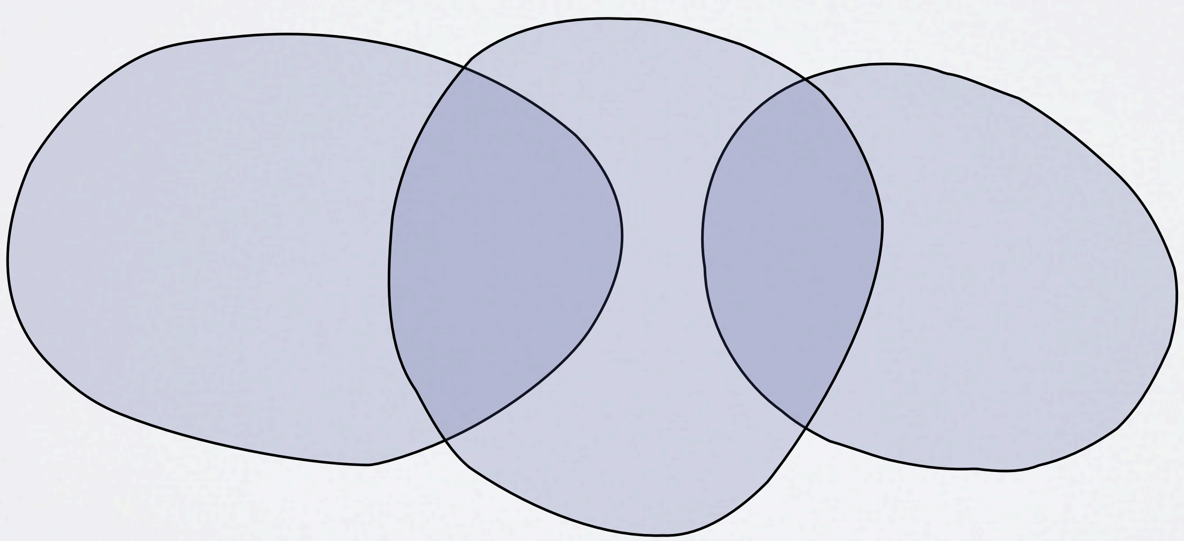
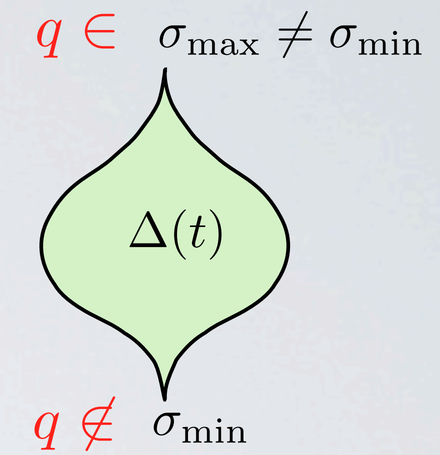


Steps for proving that collapses

$K(t) = \text{Nerve}\{D_p(t) \mid p \in P\}$

$\Delta(t) =$ set of simplices that disappear at time t



Does the operation that removes $\Delta(t)$ from $K(t)$ a collapse?

(1) Generically, $\Delta(t)$ has a unique minimal element $\sigma_{\min}$

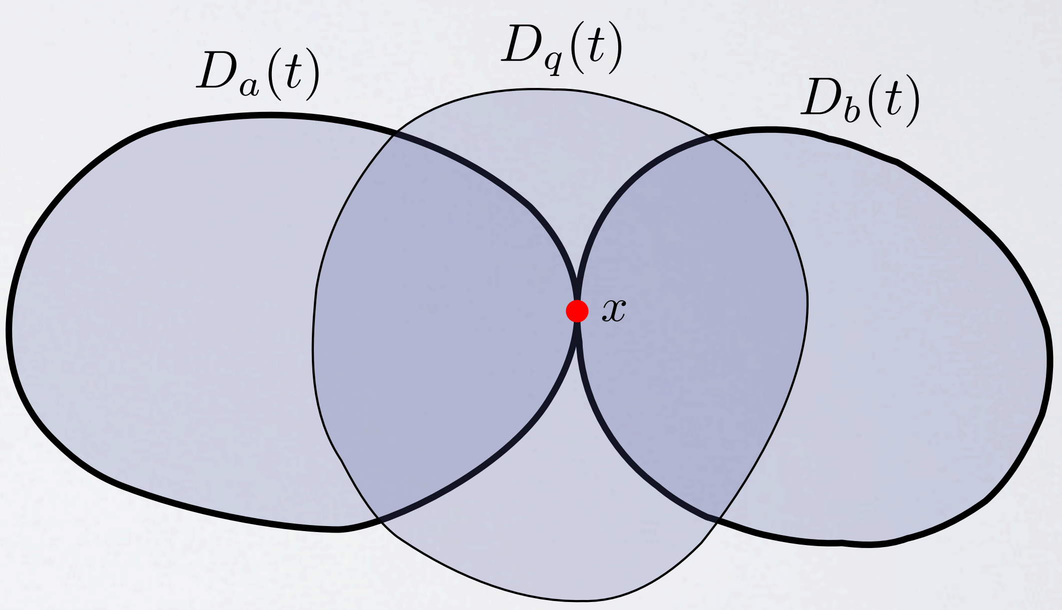
(2) $\bigcap_{p \in \sigma_{\min}} D_p(t) = \{x\}$

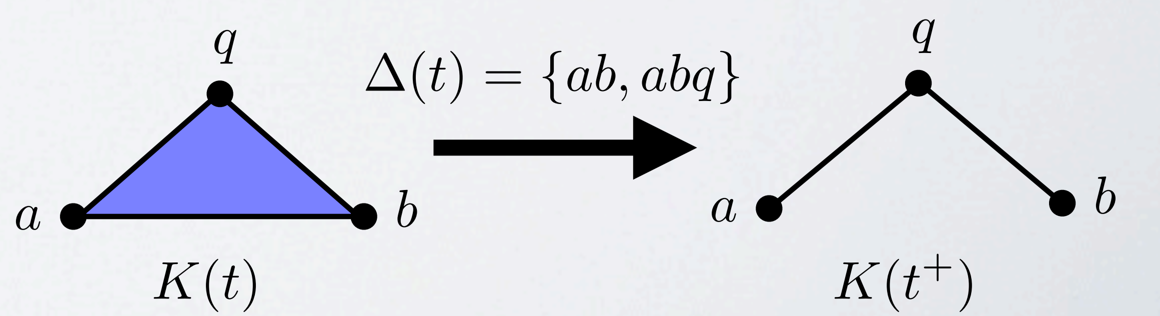
(3) $\sigma_{\max} = \{p \in P \mid x \in D_p(t)\}$

(4) $x \in \partial D_p(t), \quad \forall p \in \sigma_{\min}$

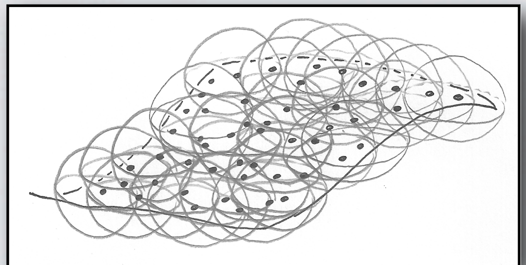
(5) $\exists q \in P$ such that $x \in D_q(t)^\circ$

(6) $\sigma_{\min} \neq \sigma_{\max} \implies$ removing $\Delta(t)$ is a collapse

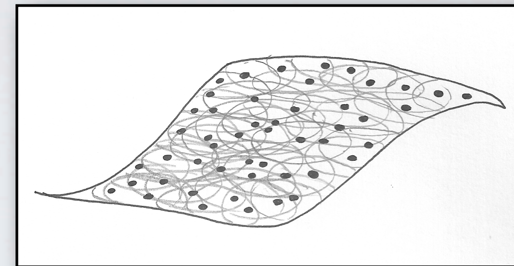




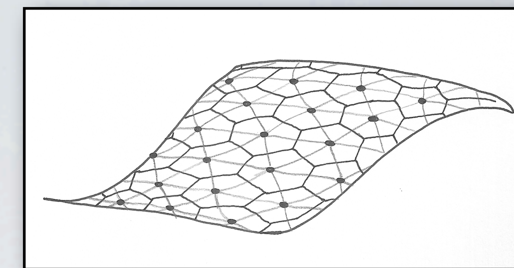
Summary



Nerve$\{B(p,\alpha) \mid p \in P\}$



Nerve$\{A \cap B(p,\alpha) \mid p \in P\}$



Nerve$\{A \cap \mathrm{Hull}_\alpha(C_v) \mid v \in V\}$

$$\mathrm{Rips}(P,\alpha) \xrightarrow{\text{collapses}} \mathrm{Cech}(P,\alpha) \xrightarrow[\text{(SC1)}]{\text{collapses}} \mathrm{Cech}_A(P,\alpha) \xrightarrow[\alpha < \text{Reach } A]{\text{collapses}} \text{Nerve of an } \alpha\text{-robust covering of } A$$

(SC2) $d_H(A,P) \le \varepsilon < \frac{2\sqrt{2-\sqrt{2}}-\sqrt{2}}{2+\sqrt{2}}\,\mathrm{Reach}(A)$

$\alpha \approx 7.22\varepsilon$

[ALS12b]

$\mathrm{Cech}(P,\alpha) \simeq P^{\oplus\alpha}$ Nerve Lemma

$\mathrm{Cech}_A(P,\alpha) \simeq \text{Shape } A$ Nerve Lemma

$\alpha < \text{Reach } A$ $\quad A \subset P^{\oplus\alpha}$

$$P^{\oplus\alpha} \xrightarrow{\text{deform. retracts}} \text{Shape } A$$

Shapes for which
$\exists$ such triangulations ?

(SC1) $d_H(A,P) \le \varepsilon < (3-\sqrt{8})\,\text{Reach } A$

$\alpha = (2+\sqrt{2})\varepsilon$

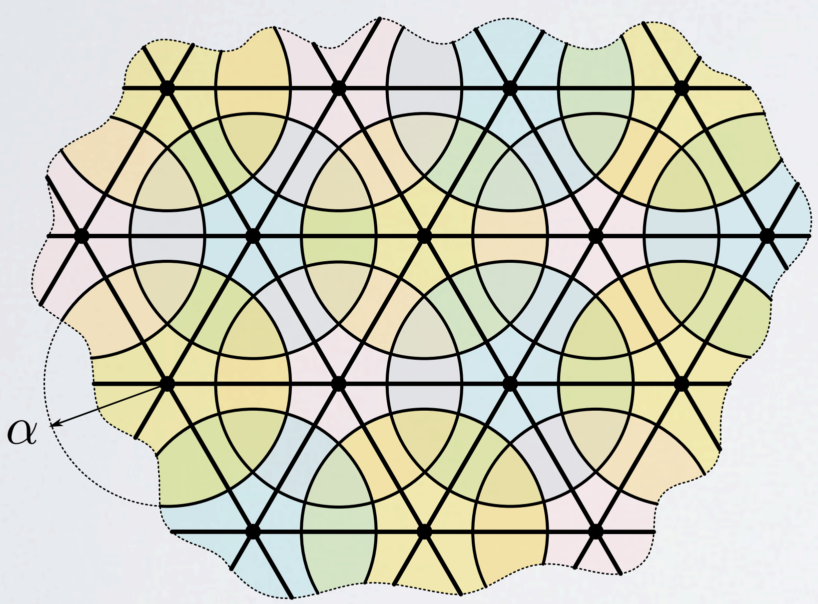
[Niyogi Smale Weinberger 2004]

α-Nice triangulations

$\mathrm{Rips}(P, \alpha)$ $\xrightarrow{\text{sequence of collapses}}$ Nerve of an α-robust covering of A

Conditions (SC2)

A triangulation of A is α-*nice* if nerve of an α-robust covering of A



$\mathbb{R}^2$

T = triangulation of $\mathbb{R}^2$ with equilateral triangles

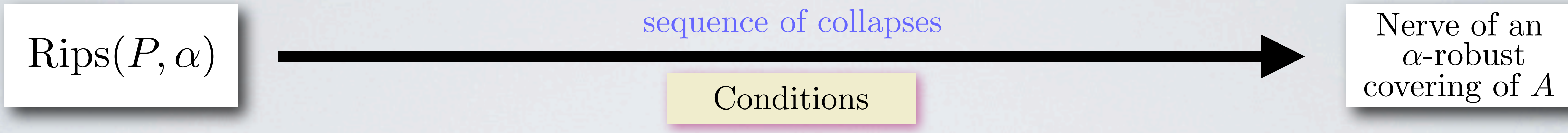
$\mathcal{C} = \{B(v, \alpha) \mid v \in \mathrm{Vertices}(T)\} \quad \setminus \quad B(v, \alpha) \subset \mathrm{St}_T(v)$

Then, $T = \mathrm{Nerve}(\mathcal{C})$

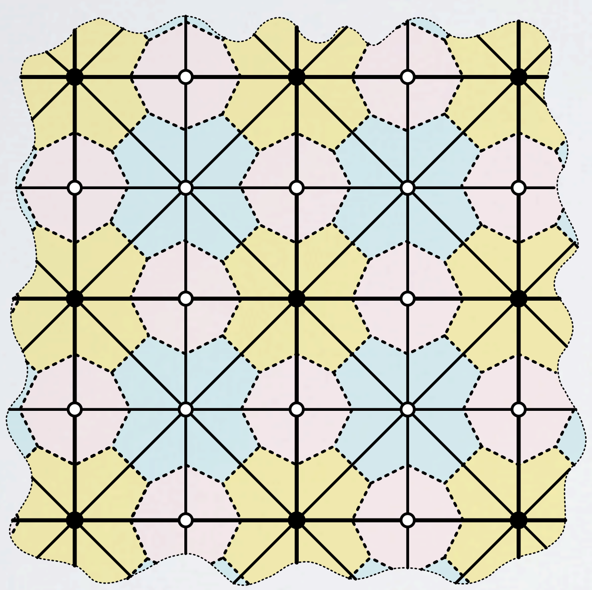
$\mathcal{C}$: α-robust

T : α-nice

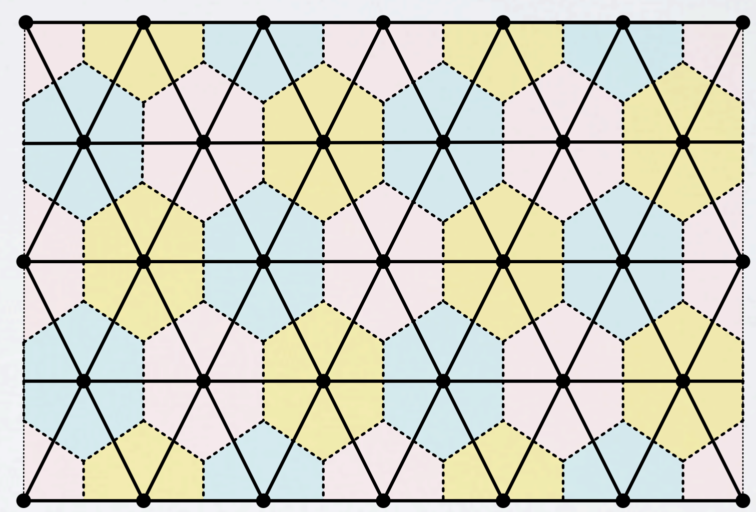
Nicely triangulable spaces



A space is "nicely triangulable" if it has an α-nice triangulation for all α



$\mathbb{R}^m$



The flat torus $\mathbb{T}^2 \subset \mathbb{R}^4$

Can we find other spaces that are "nicely triangulable"?

Can we turn all this into a practical algorithm?