Birth and death in discrete Morse theory

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joint work with

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ACAT Bremen 2013

Motivation

Our goal is to present a discrete version of the following smooth phenomenon.

As t varies, the critical points of f_t move around, at certain values of t new critical points are born or die.

Note: births and deaths happen in pairs, at finitely many parameter values $t_i, i = 1, \ldots, k$, where f_{t_i} is not Morse

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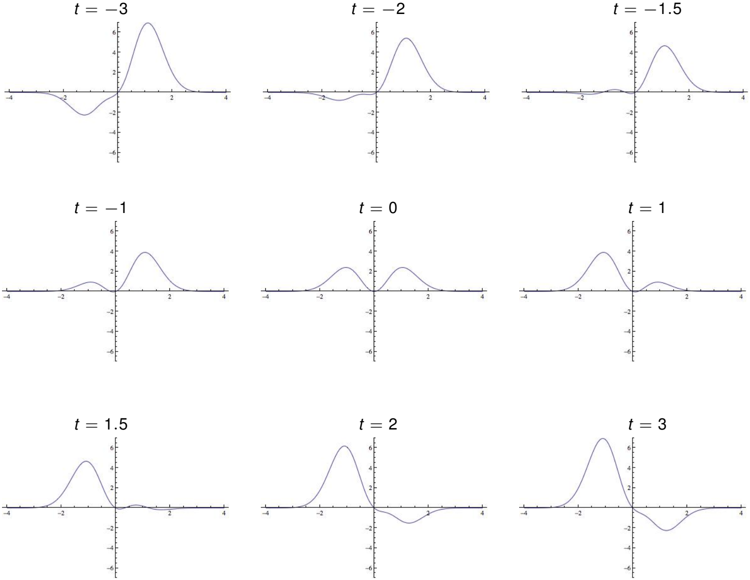
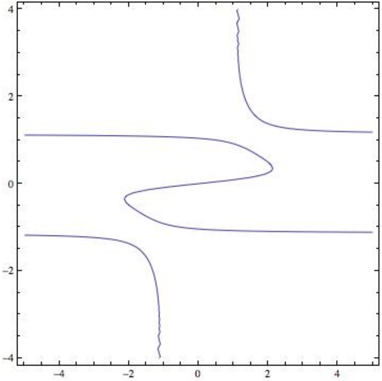
Example

$f : \mathbb{R} \times I \to \mathbb{R}$, $f_t(x) = e^{-x^2}(x^4/2 - 3tx^3 + 6x^2 - tx)$



Bifurcation diagram

This births and deaths are reflected in the bifurcation diagram in the (t, x) plane.



$t \rightarrow$

In this case it is simply the curve $f_x(x, t) = 0$

Discrete case

In the discrete case, a sample of function values of $f_t : M \to \mathbb{R}$ is given for each $t = t_0, \dots, t_r$.

A discrete version of Morse theory is used to identify the critical values for each f_{t_i},

and we would like to have a discrete version of the bifurcation diagram to trace the critical values.

Discrete and PL versions of Morse theory

This is by no means a complete list . . .

1. Computational Morse theory
   - in 2D: Herbert Edelsbrunner, John Harer, and Afra Zomorodian, *Hierarchical Morse com- plexes for piecewise linear 2-manifolds* (2001),
   - in 3D: Herbert Edelsbrunner, John Harer, Vijay Natarajan, and Valerio Pascucci, *Morse- Smale complexes for piecewise linear 3-manifolds* (2003),
2. PL Morse theory
   - T.F. Banchoff, *Critical points and curvature for embedded polyhedra* (1967),
   - U . Brehmand, W . Kühnel, *Combinatorial manifolds with few vertices* (1987) ,
   - Mladen Bestvina and Noel Brady, *Morse theory and finiteness properties of groups* (1997), Betsvina, *PL Morse theory*, published as notes in several places
3. Robin Forman's Discrete Morse theory (1990's)

Discrete Morse functions

Forman (1995,1998)

M a regular cell complex

A *discrete Morse function* $F : M \to \mathbb{R}$ associates a value to each cell $\sigma \in M$ such that on every closed cell F is striclty increasing with respect to dimension, except possibly in one direction.

That is, for every $\sigma^k \in M$

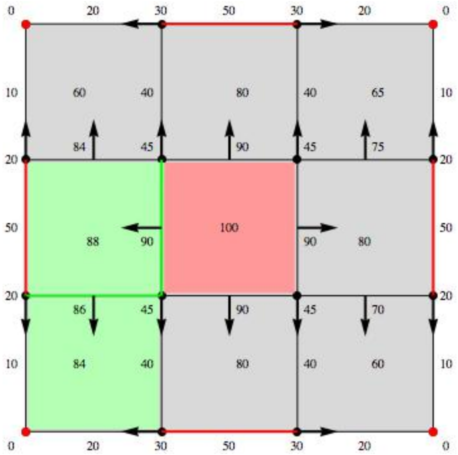
- $F(\tau^{k-1}) \geq F(\sigma^k)$ for at most one face $\tau < \sigma$ or
- $F(\tau^{k+1}) \leq F(\sigma^k)$ for at most one coface $\tau > \sigma$.

The pairs (σ^k, τ^{k+1}), $F(\sigma) \geq F(\tau)$, contain the *regular cells*.

The unpaired cells are *critical* with index equal to the dimension of the cell.

Gradient paths

A *gradient path* is a sequence of cells in adjacent dimensions forming a path along which the function values decrease.



A *discrete vector field* on a regular cell complex M is a partial pairing of its cells, consisting of mutually disjoint pairs

$$V = \{(\sigma^k, \tau^{k+1}) \mid \sigma < \tau\},$$

For example, the regular pairs (σ^k, τ^{k+1}) of a discrete Morse function on M form the *discrete gradient vector field*.

Theorem (Forman): A discrete vector field on a finite regular cell complex M is the gradient field of some discrete Morse function if and only if it has no closed paths.

True also for infinite complexes (Ayala, Jerše, M, Vilches, 2011).

Parametric case

M a regular cell complex

$F_{t_i} : M \to \mathbb{R}$, $0 < t_1 < \cdots < t_r = 1$ discrete Morse functions.

We wish to study how the critical cells move in M as i varies and construct bifurcation diagrams similar to the smooth case.

Possibly the triangulations of M can be different for each i:

$F_{t_i} : M_i \to \mathbb{R}$, $0 < t_1 < \cdots < t_r = 1$.

Vines and vineyards

Cohen-Steiner, Edelsbrunner, Morozov (2006)

Instead of discrete Morse functions, *monotonic functions* on a simplicial complex are considered: these are fucntions which are nondecresing on increasing chains of faces (similar but less restrictive than discrete Morse functions, on simplicial complexes).

If f_t is a 1-parametric family of such functions, then *vines* are the connected curves in the plane, obtained by tracing off-diagonal points in the sublevel persistece diagrams.

The collection of vines is a *vineyard* - it encodes the bifurcation diagram of such a 1-parametric family.

Connecting critical cells

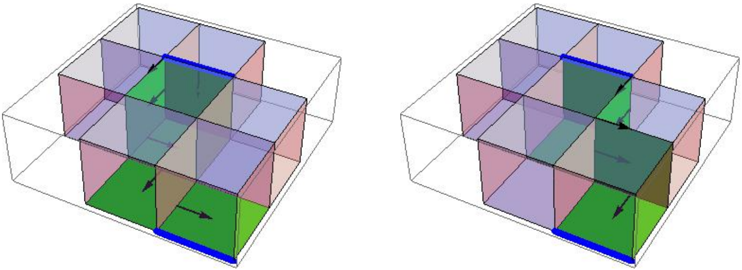
$F_{t_i} : M \to \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_r = 1$, discrete Morse fuctions on a regular cell complex M, V_i the corresponding gradient vector fields.

$\alpha^k \in M$ critical in V_i and $\beta^k \in M$ critical in V_{i-1}.

Then α is connected to β if there exists a cell γ^k, and a path of $k, k-1$ cells in V_i from α to γ, and a path of $k+1, k$ cells in V_{i-1} from γ to β.

The algorithm mimicks a discrete Morse function on $M \times [0, 1]$ which increases with t and up to a constant coincides with F_{t_i} on the slice $M \times \{t_i\}$ for all i.

Connecting critical cells



On "nice" cell complexes: the discrete vector field V_i is extended to $M \times [t_{i-1}, t_i]$ by connecting α^k to the product $\alpha^k \times [t_{i-1}, t_i]$ and following V_i along regular pairs.

Cells are connected along a path in the extended vector field.

Discrete bifurcation diagram

$\alpha^k \in M$ (critical in V_i) is *strongly connected* to $\beta^k \in M$ (critical in V_{i-1}) if there exists

- a cell γ'^k, and a path of $k, k-1$ cells in V_{i-1} from α to γ', and a path of $k, k+1$ cells in V_i from γ' to β,
- a cell γ^k, and a path of $k, k-1$ cells in V_i from β to γ, and a path of $k, k+1$ cells in V_{i-1} from γ to α

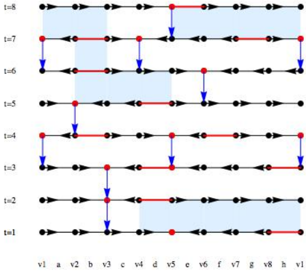
Strong connections between critical cells give the *discrete bifurcation diagram*.

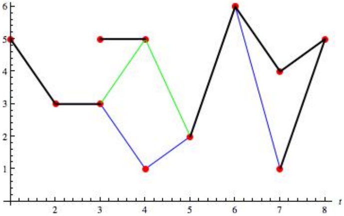
A critical cell α in V_i

- *is born at t_i* if it is not strongly connected to any critical cell at t_{i-1}
- *dies at t_{i+1}* if it is not strongly connected to any critical cell at t_{i+1}

Example

A sequence of 8 discrete vector fields on a circle and the bifurcation diagram for dim 0 critical cells:





Parametric discrete Morse functions II

What if the cell decompositions of M are different for each time slice t_i?

Important in applications, since the sample points may change over time.

$F_{t_i} : M_i \to \mathbb{R}$, $0 = t_0 < t_1 < \cdots < t_r = 1$ discrete Morse functions on a regular cell complex M.

We assume that M_i and M_{i+1} have a common subdivision $M_{i+1/2}$.

Refinements

A cell complex N is a *refinement* of a cell complex M if each cell of N is contained in a cell of M.

Let $g\colon N \to M$ associate to each cell $\sigma \in N$ the smallest cell in M containing σ.

A discrete vector field W on N is a *refinement* of a discrete vector field V on M if there exists a choice $h\colon M \to N$ which

- maps an M-cell to a smaller N-cell contained in it,
- maps critical M-cells to critical N-cells and regular M-pairs to regular N-pairs,
- the N-pairs in W which are not inherited from V are restricted to interiors of M-cells.

A path in the refinement W consists of an initial part contained inside a single M-cell, then moves through the cells of a V-path in M one by one.

Under relatively mild restrictions on the cell complexes M and N there exists a refinement W of V with no additional critical cells.

These restrictions are:

- N is obtained from M by a sequence of two types of subdivisions:
  - adding a new vertex in the interior of a cell and subdividing the cell into cones to the boundary cells,
  - dividing a cell along a sphere of codimension 1 which forms a subcomplex in its boundary into three new cells,
- the boundary sphere of each cell admits a discrete vector field with the minimal number of critical cells (that is, 2).

Connecting critical cells

$F_{t_i} : M_i \to \mathbb{R}$ a finite family of discrete Morse functions and M_i possibly different triangulations,

$M_{i-1/2}$ a common subdivision of M_i and M_{i-1}

V_{i^-} a refinement of V_i on $M_{i-1/2}$ with respect to the choice $h\colon M_i \to M_{i-1/2}$

Then: a critical cell α^k in M_i is connected to a critical cell β^k in M_{i-1} if there exists a cell γ^k in $M_{i-1/2}$ which is connected to $h_i(\alpha)$ along a path in V_{i^-} and to β along a path in V_{i-1}.

This can be done also backwards, from M_{i-1} to M_i. Critical cells are strongly connected if they are connected both ways.

Discrete approximations of functions

X a compact smooth manifold $f : M \to \mathbb{R}$ a smooth Morse function, v the corresponding gradient field.

Gallais (2010): There exist a C^1-triangulation M of X and a discrete gradient field V on M such that critical cells of V are in correspondence with the critical points of f, each critical point lies in the interior of its critical cell, and V-paths between critical cells correposnd to integral curves of v.

Benedetti, *Smoothing discrete Morse theory* (2013):

For any C^1 triangulation of a compact smooth manifold M (possibly with boundary) with a smooth Morse function, there exists a barycentric subdivision and a discrete Morse function on it such that the critical cells are in bijective correspondence with the critical points.

Parametric discrete approximation

$f : X \times I \to \mathbb{R}$ a family of smooth functions which are Morse at parameters $0 = t_0 < t_1 < \cdots < t_r = 1$, v_t the corresponding gradient fields,

M_i cell decompositions of X and V_i discrete gradient fields on M_i like above.

Then there exist subdivisions $M_{i\pm 1/2}$ and refinements $V^\pm$ with no excess critical points such that the algorithm will strongly connect two critical cells corresponding to connected critical points in the bifurcation diagram of f.

Applications to data

The critical cells of a discrete Morse function can represent interesting features in the data which we would like to track.

In order to do this we usually have to first

- find a cell decomposition of the space from which the data comes (in some cases, for example in images, this is given)
- extend the function values on the vertices to a discrete Morse function on the cell complex with as few critical cells as is suitable to model the data (Knudson, King, M. *Generating discrete Morse functions from data*, 2005)

Used for tracinging features (voids, canals) in CT scans, track moving figures, qualitative machine learning

Cancelling critical cells

In order to eliminate excess critical cells (due to noise or too small details) we use cancelling:

If a $k+1$-critical cell α is connected to a k-critical cell β along precisely one gradient path starting in its boundary, then, by reversing all arrows along the path, pairing α with the initial cell of the path and β with the final cell, two critical cells are eliminated.

Since no close paths are generated, this produces a new gradient vector field.

Cancelling can be done between cells with vales differing by less then some predefined threshold p. Some care has to be taken to that the cancelling is performed uniformly in neighbouring slices.

Tracing the skier

Thank you!