Spaces of directed paths as simplicial complexes

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Applied and Computational Algebraic Topology

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Agenda

- Examples: State spaces and associated path spaces in Higher Dimensional Automata (HDA)
- Motivation: from Concurrency Theory
- Simplest case: State spaces and path spaces related to linear PV-programs
- Tool: Cutting up path spaces into contractible subspaces
- Homotopy type of path space described by a matrix poset category and realized by a prodsimplicial complex
- Algorithmics: Detecting dead and alive subcomplexes/matrices
- Outlook: How to handle general HDA – with directed loops
- Case: Directed loops on a punctured torus (joint with K. Ziemiański, Warsaw)

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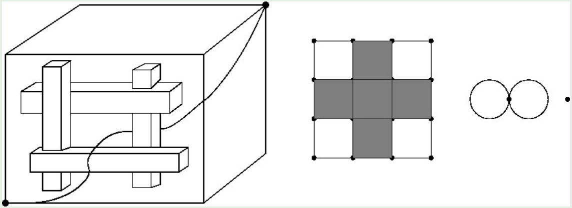
Agenda

Intro: State space, directed paths and trace space
Problem: How are they related?

Example 1: State space and trace space for a semaphore HDA



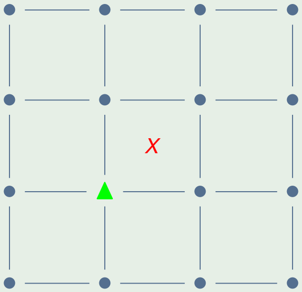
State space:
a 3D cube $\vec{I}^3 \setminus F$
minus 4 box obstructions
pairwise connected

Path space model contained in torus $(\partial\Delta^2)^2$ –
homotopy equivalent to a wedge of two circles and a point: $(S^1 \vee S^1) \sqcup *$

Analogy in standard algebraic topology

Relation between space X and loop space ΩX.

Example 2: Punctured torus



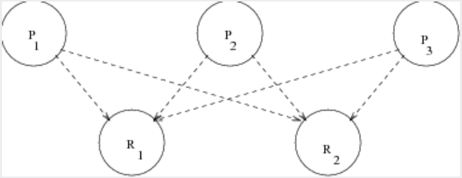
State space: Punctured torus X and branch point ▲:
2D torus $\partial\Delta^2 \times \partial\Delta^2$ with a rectangle $\Delta^1 \times \Delta^1$ removed

Path space model:
Discrete infinite space of dimension 0 corresponding to $\{r, u\}^*$.

Question: Path space for a punctured torus in higher dimensions?
Joint work with K. Ziemiański.

(Mutual) Exclusion

occurs, when n processes P_i compete for m resources R_j.



Only k processes can be served at any given time.

Semaphores: A simple model for (mutual) exclusion

Semantics: A processor has to lock a resource and to relinquish the lock later on!

Description/abstraction: $P_i : \dots PR_j \dots VR_j \dots$ (E.W. Dijkstra)

P: probeer; V: verhoog

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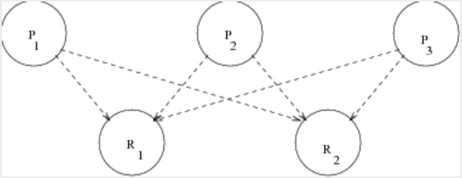

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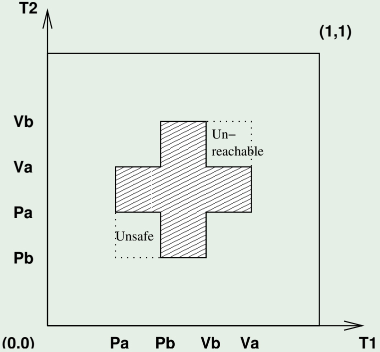

PV-diagram from

$P_1 : P_a P_b V_b V_a$

$P_2 : P_b P_a V_a V_b$

Executions are directed paths – since time flow is irreversible – avoiding a forbidden region (shaded). Dipaths that are dihomotopic (through a 1-parameter deformation consisting of dipaths) correspond to equivalent executions. Deadlocks, unsafe and unreachable regions may occur.

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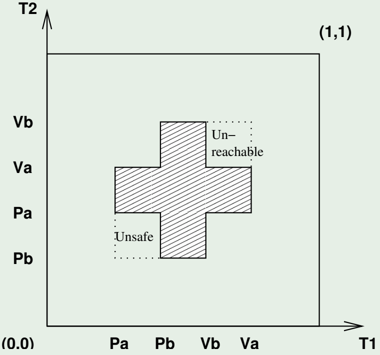

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A geometric model: Schedules in "progress graphs"

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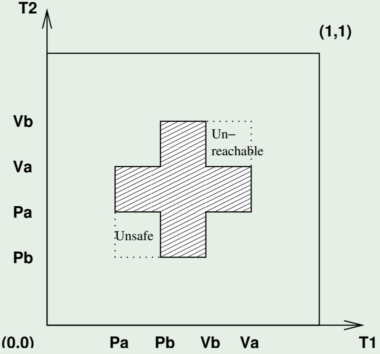

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The state space

A linear PV-program is modeled as the complement of a forbidden region F consisting of a number of holes in an n-cube:

- Hole = isothetic hyperrectangle $R^i =]a_1^i, b_1^i[\times \cdots \times]a_n^i, b_n^i[\subset I^n, 1 \leq i \leq l$: with minimal vertex $\mathbf{a}^i$ and maximal vertex $\mathbf{b}^i$.
- State space $X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^l R^i$
  X inherits a partial order from $\vec{I}^n$.
  d-paths are order preserving.

More general concurrent programs $\rightsquigarrow$ HDA

Higher Dimensional Automata (HDA, V. Pratt; 1990):

- Cubical complexes: like simplicial complexes, with (partially ordered) hypercubes instead of simplices as building blocks.[a]
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Definition

- X a d-space[a], $a, b \in X$.
  $p : \vec{I} \to X$ a d-path in X (continuous and "order-preserving") from a to b.
- $\vec{P}(X)(a,b) = \{p : \vec{I} \to X \mid p(0) = a, p(b) = 1, p \text{ a d-path}\}$.
  Trace space $\vec{T}(X)(a,b) = \vec{P}(X)(a,b)$ modulo increasing reparametrizations.
  In most cases: $\vec{P}(X)(a,b) \simeq \vec{T}(X)(a,b)$.
- A dihomotopy in $\vec{P}(X)(a,b)$ is a map $H : \vec{I} \times I \to X$ such that $H_t \in \vec{P}(X)(a,b),\ t \in I$; ie a path in $\vec{P}(X)(a,b)$.

[a] Marco Grandis

Aim:

Description of the homotopy type of $\vec{P}(X)(a,b)$ as explicit finite dimensional (prod-)simplicial complex.
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Tool: Subspaces of state space X and of $\vec{P}(X)(\mathbf{0},\mathbf{1})$

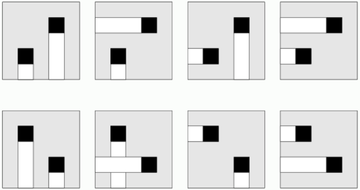
$X = \vec{I}^n \setminus F,\ F = \bigcup_{i=1}^{l} R^i;\ R^i = [\mathbf{a}^i, \mathbf{b}^i];\ \mathbf{0}, \mathbf{1}$ the two corners in I^n.

Definition

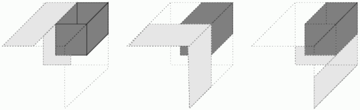
1. $X_{ij} = \{x \in X \mid x \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j\}$ – *direction j restricted at hole i*
2. M a binary $l \times n$-matrix: $X_M = \bigcap_{m_{ij}=1} X_{ij}$ – *Which directions are restricted at which hole?*

Examples: two holes in 2D – one hole in 3D (dark)

$M =$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$



$M =$ $[100]$ $[010]$ $[001]$



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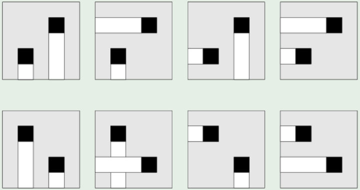

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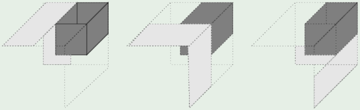

Binary matrix posets

$M_{l,n}$ poset ($\leq$) of binary $l \times n$-matrices

$M_{l,n}^{R,*}$ no row vector is the zero vector – every hole obstructed in at least one direction

A cover by contractible subspaces

Theorem

1.
$$\vec{P}(X)(\mathbf{0},\mathbf{1}) = \bigcup_{M \in M_{l,n}^{R,*}} \vec{P}(X_M)(\mathbf{0},\mathbf{1}).$$

2. *Every path space $\vec{P}(X_M)(\mathbf{0},\mathbf{1})$, $M \in M_{l,n}^{R,*}$, is empty or contractible.* *Which is which? Deadlocks!*
3. *New: Modification leads to fewer and smaller "patches" with fewer intersections!*

Proof.

(2) Subspaces X_M, $M \in M_{l,n}^{R,*}$ are closed under $\vee$ = l.u.b. □

Covers by contractible (or empty) subspaces

Bookkeeping with binary matrices

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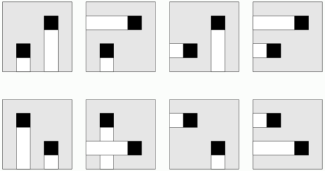
Combinatorics
poset category
$\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \subseteq M_{l,n}^{R,*} \subseteq M_{l,n}$
$M \in \mathcal{C}(X)(\mathbf{0},\mathbf{1})$ "alive"

Topology:
prodsimplicial complex
$\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \subseteq (\Delta^{n-1})^l$
$\Delta_M = \Delta_{m_1} \times \cdots \times \Delta_{m_l} \subseteq$
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ – one simplex Δ_{m_i} for every hole

$$\Leftrightarrow \vec{P}(X_M)(\mathbf{0},\mathbf{1}) \neq \emptyset.$$

New: Modified definitions gives rise to a "smaller" simplicial complex, in particular of far lower dimension!

Examples of path spaces



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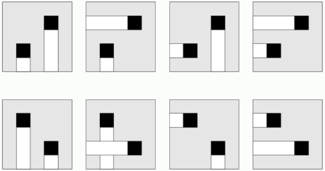

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A combinatorial model and its geometric realization
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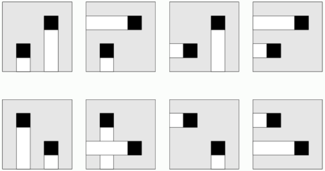

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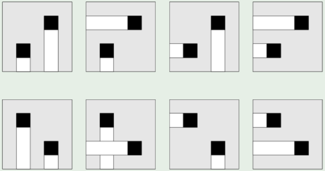

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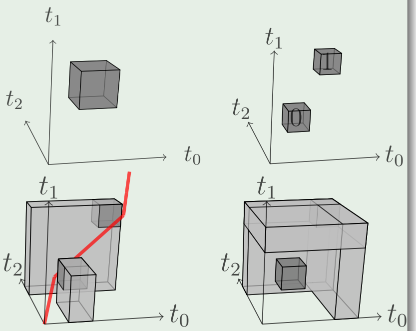
Further examples

State spaces, "alive" matrices and path spaces

1. $X = \vec{I}^n \setminus \vec{J}^n$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^{R,*}_{1,n} \setminus \{[1,\ldots,1]\}$.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) = \partial\Delta^{n-1} \simeq S^{n-2}$.
2. $X = \vec{I}^n \setminus (\vec{J}^n_0 \cup \vec{J}^n_1)$
   - $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) = M^{R,*}_{2,n} \setminus$ matrices with a $[1,\ldots,1]$-row.
   - $\mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq S^{n-2} \times S^{n-2}$.



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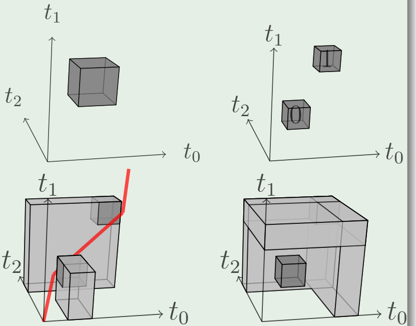

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Homotopy equivalence between path space $\vec{P}(X)(\mathbf{0},\mathbf{1})$ and prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem (A variant of the nerve lemma)

$\vec{P}(X)(\mathbf{0},\mathbf{1}) \simeq \mathbf{T}(X)(\mathbf{0},\mathbf{1}) \simeq \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.

Proof.

- Functors $\mathcal{D},\mathcal{E},\mathcal{T} : \mathcal{C}(X)(\mathbf{0},\mathbf{1})^{(\mathrm{op})} \to \mathbf{Top}$:
  $\mathcal{D}(M) = \vec{P}(X_M)(\mathbf{0},\mathbf{1})$,
  $\mathcal{E}(M) = \Delta_M$,
  $\mathcal{T}(M) = *$
- $\mathrm{colim}\ \mathcal{D} = \vec{P}(X)(\mathbf{0},\mathbf{1})$, $\mathrm{colim}\ \mathcal{E} = \mathbf{T}(X)(\mathbf{0},\mathbf{1})$,
  $\mathrm{hocolim}\ \mathcal{T} = \Delta\mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
- The trivial natural transformations $\mathcal{D} \Rightarrow \mathcal{T}, \mathcal{E} \Rightarrow \mathcal{T}$ yield:
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□

Homotopy equivalence between path space $\vec{P}(X)(\mathbf{0},\mathbf{1})$ and prodsimplicial complex $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Theorem (A variant of the nerve lemma)

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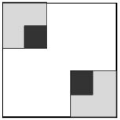
Detection of dead and alive matrices & subcomplexes

An algorithm starts with deadlocks and unsafe regions!

Allow less = forbid more!

Remove **extended** hyperrectangles R^i_j
$:= [0, b^i_1[\times \cdots \times [0, b^i_{j-1}[\times]a^i_j, 1] \times [0, b^i_{j+1}[\times \cdots \times [0, b^i_n[\supset R^i$.



$$X_M = X \setminus \bigcup_{m_{ij}=1} R^i_j.$$

New: Further extension of the R^i_j "covering" far more obstruction hyperrectangles.

Theorem

The following are equivalent:

1. $\vec{P}(X_M)(\mathbf{0},\mathbf{1}) = \emptyset \Leftrightarrow M \notin \mathcal{C}(X)(\mathbf{0},\mathbf{1})$.
2. There is a "dead" matrix $N \leq M$, $N \in M^{C,u}_{l,n}$ such that $\bigcap_{n_{ij}=1} R^i_j \neq \emptyset$ – giving rise to a deadlock unavoidable from $\mathbf{0}$, i.e., $T(X_N)(\mathbf{0},\mathbf{1}) = \emptyset$.
   $M^{C,u}_{l,n}$: every column a unit vector – every direction obstructed once.

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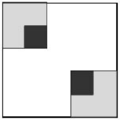

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From $\mathcal{C}(X)(\mathbf{0},\mathbf{1})$ to properties of path space

Questions answered by homology calculations using $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$

Questions

- Is $\vec{P}(X)(\mathbf{0},\mathbf{1})$ path-connected, i.e., are all (execution) d-paths dihomotopic (lead to the same result)?
- Determination of path-components?
- Are components simply connected?
  Other topological properties?

Strategies – Attempts

- Implementation of $\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ in ALCOOL at CEA/LIX-lab.: Goubault, Haucourt, Mimram
- The prodsimplicial structure on $\mathcal{C}(X)(\mathbf{0},\mathbf{1}) \leftrightarrow \mathbf{T}(X)(\mathbf{0},\mathbf{1})$ leads to an associated chain complex of vector spaces over a field.
- Use fast algorithms (eg Mrozek's CrHom etc) to calculate the homology groups of these chain complexes even for quite big complexes: M. Juda (Krakow).
- Number of path-components: $rkH_0(\mathbf{T}(X)(\mathbf{0},\mathbf{1}))$.
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Open problem: Huge complexes – complexity

Huge prodsimplicial complexes

l obstructions, n processors:
$\mathbf{T}(X)(\mathbf{0},\mathbf{1})$ is a subcomplex of $(\partial\Delta^{n-1})^l$:
potentially a huge high-dimensional complex.

Possible antidotes – new

- $\tilde{a}^i_j :== \max\{-1, a^{i'}_j \mid a^{i'}_j < a^i_j, \mathbf{b}^i_{\hat{j}} \leq \mathbf{b}^{i'}_{\hat{j}}\}$.
  Replace X_{ij} by
  $Y_{ij} := \{\mathbf{x} \in X \mid (\mathbf{x} \leq \mathbf{b}^i \Rightarrow x_j \leq a^i_j) \wedge (x_j \leq \tilde{a}^i_j \Rightarrow \mathbf{x}_{\hat{j}} \leq \mathbf{b}^i_{\hat{j}})\}$
  Complements S_{ij} of Y_{ij} are unions of hyperrectangles.
- Vertices of s. cx.: Collections $\mathcal{S}$ of S_{ij} such that
  - $F = \bigcup_i R^i \subset \bigcup_{S_{ij}\in\mathcal{S}} S_{ij}$;
  - Every $R^{i'}$ is contained in exactly one $S_{ij} \in \mathcal{S}$;
  - $\vec{P}(\vec{I}^n \setminus \bigcup_{S_{ij}\in\mathcal{S}} S_{ij})(\mathbf{0},\mathbf{1}) \neq \emptyset$.

  Recursive generation of vertices and s. cx.
- Observation: Two intersecting obstructions (in I^n) can at most contribute to the diagonal $\partial\Delta^{n-1} \hookrightarrow \partial\Delta^{n-1} \times \partial\Delta^{n-1}$. Similar for a chain of intersecting obstructions.

Open problems: Variation of end points

Conncection to MD persistence?

Components?!

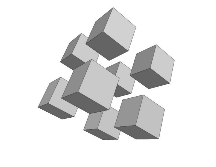
- So far: $\vec{T}(X)(\mathbf{0}, \mathbf{1})$ - fixed end points.
- Now: Variation of $\vec{T}(X)(\mathbf{a}, \mathbf{b})$ of start and end point, giving rise to filtrations.
- At which thresholds do homotopy types change?
- How to cut up $X \times X$ into components so that the homotopy type of trace spaces with end point pair in a component is invariant?
- Birth and death of homology classes?
- Compare with multidimensional persistence (Carlsson, Zomorodian).

Case: d-paths on a punctured torus

with directed loops!

Punctured torus and n-space

n-torus $T^n = \mathbf{R}^n/_{\mathbf{Z}^n}$.
forbidden region $F^n = ([\frac{1}{4}, \frac{3}{4}]^n + \mathbf{Z}^n)/_{\mathbf{Z}^n} \subset T^n$.
punctured torus $Q^n = T^n \setminus F^n \simeq T^n_{(n-1)}$ (skel.)
punctured n-space $\tilde{Q}^n = \mathbf{R}^n \setminus ([\frac{1}{4}, \frac{3}{4}]^n + \mathbf{Z}^n) \simeq \mathbf{R}^n_{(n-1)}$
with d-paths via quotient map $\mathbf{R}^n \downarrow T^n$.



Aim: Describe the homotopy type of loops $\vec{P}(Q) = \vec{P}(Q)(\mathbf{0}, \mathbf{0})$

$\vec{P}(Q) \hookrightarrow \Omega Q(\mathbf{0}, \mathbf{0}) \rightsquigarrow$ disjoint union $\vec{P}(Q) = \bigsqcup_{\mathbf{k} \geq \mathbf{0}} \vec{P}(\mathbf{k})(Q)$
with multiindex = multidegree $\mathbf{k} = (k_1, \dots, k_n) \in \mathbf{Z}^n_+, k_i \geq 0$.
$\vec{P}(\mathbf{k})(Q) \cong \vec{P}(\tilde{Q}^n)(\mathbf{0}, \mathbf{k}) =: Z(\mathbf{k})$.

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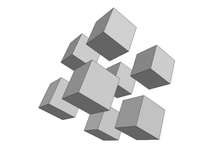

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Path spaces as colimits

Category $\mathcal{J}(n)$

Poset category of proper non-empty subsets of $[1:n]$ with inclusions as morphisms.
Via characteristic functions isomorphic to the category of non-identical bit sequences of length n: $\varepsilon = (\varepsilon_1, \ldots \varepsilon_n) \in \mathcal{J}(n)$.
$B\mathcal{J}(n) \cong \partial\Delta^{n-1} \cong S^{n-2}$.

Definition

$U_\varepsilon(\mathbf{k}) := \{\mathbf{x} \in \mathbf{R}^n | \ \varepsilon_j = 1 \Rightarrow x_j \leq k_j - 1 \text{ or } \exists i : x_i \geq k_i\}$
$Z_\varepsilon(\mathbf{k}) := \vec{P}(U_\varepsilon(\mathbf{k}))(\mathbf{0}, \mathbf{k})$.

Lemma

$Z_\varepsilon(\mathbf{k}) \simeq Z(\mathbf{k} - \varepsilon)$.

Theorem

$Z(\mathbf{k}) = \operatorname{colim}_{\varepsilon \in \mathcal{J}(n)} Z_\varepsilon(\mathbf{k}) \simeq \operatorname{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z_\varepsilon(\mathbf{k}) \simeq \operatorname{hocolim}_{\varepsilon \in \mathcal{J}(n)} Z(\mathbf{k} - \varepsilon)$.

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By construction $\mathbf{k} \leq \mathbf{l} \Rightarrow X(\mathbf{k}) \subseteq X(\mathbf{l})$; $X(\mathbf{1}) \cong \partial \Delta^{n-1}$.

Inductive homotopy equivalences

$q(\mathbf{k}) : Z(\mathbf{k}) \to X(\mathbf{k})$:

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Homology and cohomology of space $Z(\mathbf{k})$ of d-paths

Definition

- $\mathbf{l} \ll \mathbf{m} \in \mathbf{Z}_+^n \Leftrightarrow l_j < m_j, 1 \leq j \leq n.$
- $\mathcal{O}^n = \{(\mathbf{l}, \mathbf{m}) | \, \mathbf{l} \ll \mathbf{m} \text{ or } \mathbf{m} \ll \mathbf{l}\} \subset \mathbf{Z}_+^n \times \mathbf{Z}_+^n$ – ord. pairs
- $\mathbf{B}(\mathbf{k}) := \mathbf{Z}_+^n(\leq \mathbf{k}) \times \mathbf{Z}_+^n(\leq \mathbf{k}) \setminus \mathcal{O}^n$ – unordered pairs
- $\mathcal{I}(\mathbf{k}) :=< \mathbf{lm} | \, (\mathbf{l}, \mathbf{m}) \in \mathbf{B}(\mathbf{k}) > \leq \mathbf{Z}[\mathbf{Z}_+^n(\leq \mathbf{k})]$
  graded polynomial ring

Theorem

For $n > 2$, $H^*(Z(\mathbf{k})) = \mathbf{Z}[\mathbf{Z}_+^n(\leq \mathbf{k})]/_{\mathcal{I}(\mathbf{k})}$.
All generators have degree $n-2$.
$H_*(Z(\mathbf{k})) \cong H^*(Z(\mathbf{k}))$ as abelian groups.

Proof

(Bousfield-Kan) spectral sequence argument, using projectivity of the functor $H_* : \mathcal{J}(n) \to \mathbf{Ab}_*, \ \mathbf{k} \mapsto H_*(Z(\mathbf{k}))$.

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Cube sequences

$[\mathbf{a}^*] := [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \cdots \ll \mathbf{a}^r = \mathbf{l}] \in A^n_{r(n-2)}(\mathbf{l})$
of size $\mathbf{l} \in \mathbf{Z}^n_+$, length r and degree $r(n-2)$.
$A^n_*(*)$ the free abelian group generated by all cube sequences.
$A^n_*(\leq \mathbf{k}) := \bigoplus_{\mathbf{l} \leq \mathbf{k}} A^n_*(\mathbf{l})$.
$H_{r(n-2)}(Z(\mathbf{k})) \cong A^n_{r(n-2)}(\leq \mathbf{k})$
generated by cube sequences of length r and size $\leq \mathbf{k}$.

Betti numbers of $Z(\mathbf{k})$

Theorem

$n = 2$: $\beta_0 = \binom{k_1+k_2}{k_1}$; $\beta_j = 0,\ j > 0$;

$n > 2$: $\beta_0 = 1$, $\beta_{i(n-2)} = \prod_1^n \binom{k_j}{i}$, $\beta_j = 0$ *else.*

Corollary

1. *Small homological dimension of* $Z(\mathbf{k})$: $(\min_j k_j)(n-2)$.
2. *For* $\mathbf{k} = (k, \ldots, k)$, $\beta_i(Z(\mathbf{k})) = \beta_{k(n-2)-i}(Z(\mathbf{k}))$.

Interpretation via cube sequences
Betti numbers

Cube sequences

$[\mathbf{a}^*] := [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \cdots \ll \mathbf{a}^r = \mathbf{l}] \in A^n_{r(n-2)}(\mathbf{l})$
of size $\mathbf{l} \in \mathbf{Z}^n_+$, length r and degree $r(n-2)$.
$A^n_*(*)$ the free abelian group generated by all cube sequences.
$A^n_*(\leq \mathbf{k}) := \bigoplus_{\mathbf{l} \leq \mathbf{k}} A^n_*(\mathbf{l})$.
$H_{r(n-2)}(Z(\mathbf{k})) \cong A^n_{r(n-2)}(\leq \mathbf{k})$
generated by cube sequences of length r and size $\leq \mathbf{k}$.

Betti numbers of $Z(\mathbf{k})$

Theorem

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A slight generalization and an "explanation"

- The result can be stated and generalized for a complex $T^n_{(n-1)} \subset K \subset T^n$ – with universal cover $\mathbf{R}^n_{(n-1)} \subset \tilde{K} \subset \mathbf{R}^n$. Homology is generated by cube sequences $[\mathbf{a}^*] := [\mathbf{0} \ll \mathbf{a}^1 \ll \mathbf{a}^2 \ll \cdots \ll \mathbf{a}^r = \mathbf{l}]$ such that the cells $[\mathbf{a}^i - \mathbf{1}, \mathbf{a}^i] \not\subset \tilde{K}$.
- A cube sequence $\mathbf{a}^*$ is maximal if it is not properly contained in another cube sequence with same endpoints.
- A maximal cube sequence $\mathbf{a}^*$ gives rise to a subspace $\vec{P}(\mathbf{a}^*)(\mathbf{0}, \mathbf{k}) \subset \vec{P}(\tilde{K})(\mathbf{0}, \mathbf{k})$ – concatenation of paths on boundary of cubes $[\mathbf{a}^i - \mathbf{1}, \mathbf{a}^i]$ and contractible path spaces.
- $Y(\mathbf{k}) = \bigcup_{\mathbf{a}^*} \vec{P}(\mathbf{a}^*)(\mathbf{0}, \mathbf{k}),\ \mathbf{a}^*$ maximal. Then also $Y(\mathbf{k}) \simeq \mathrm{hocolim}_{\varepsilon \in \mathcal{J}(n)} Y(\mathbf{k} - \varepsilon)$ and $Y(\mathbf{k})$ contractible if $\prod_i k_i = 0$.
- Hence $Y(\mathbf{k}) \simeq X(\mathbf{k}) \simeq Z(\mathbf{k})$.
- $\vec{P}(\mathbf{a}^*)(\mathbf{0}, \mathbf{k}) \subset \vec{P}(\tilde{K})(\mathbf{0}, \mathbf{k})$ induces an injection $H^*(\vec{P}(\mathbf{a}^*)(\mathbf{0}, \mathbf{k})) \cong H^*((S^{n-2})^r) \to H^*(\vec{P}(\tilde{K})(\mathbf{0}, \mathbf{k}))$.

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To conclude

Conclusions and challenges

- From a (rather compact) state space model (shape of data) to a finite dimensional trace space model (represent shape).
- Calculations of invariants (Betti numbers) of path space possible for state spaces of a moderate size (measuring shape).
- Dimension of trace space model reflects not the size but the complexity of state space (number of obstructions, number of processors); still: curse of dimensionality.
- Challenge: General properties of path spaces for algorithms solving types of problems in a distributed manner?
  Connections to the work of Herlihy and Rajsbaum protocol complex etc
- Challenge: Morphisms between HDA $\rightsquigarrow$ d-maps between cubical state spaces $\rightsquigarrow$ functorial maps between trace spaces. Properties? Equivalences?

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Want to know more?

Books

- Kozlov, Combinatorial Algebraic Topology, Springer, 2008.
- Grandis, Directed Algebraic Topology, Cambridge UP, 2009.

Articles

- MR, Simplicial models for trace spaces, AGT **10** (2010), 1683 – 1714.
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- Fajstrup, Trace spaces of directed tori with rectangular holes, Aalborg University Research Report R-2011-08.
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