A bridge between continuous and discrete multiD persistence

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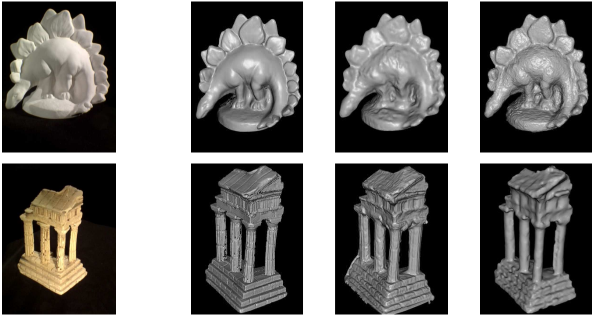
Motivation

Real object Models



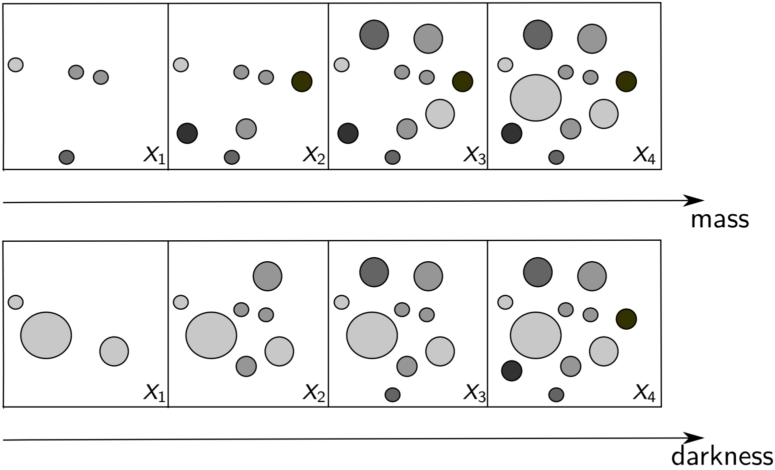
- How accurately does rank invariant comparison on discrete models approximate that on continuous objects?
- To which extent can data resolution be coarsened in order to maintain a certain error threshold on rank invariants comparison?

Outline

- Multidimensional persistence of a filtration
  - sub-level set filtrations
  - simplicial complex filtrations
- From discrete to continuous filtrations:
  - an obstacle: topological aliasing
  - a way round: axis-wise linear interpolation
- From continuous to discrete:
  - stable comparison of multi-D persistence
- Application:
  - a procedure to predetermine the model precision required to reach a given error threshold.

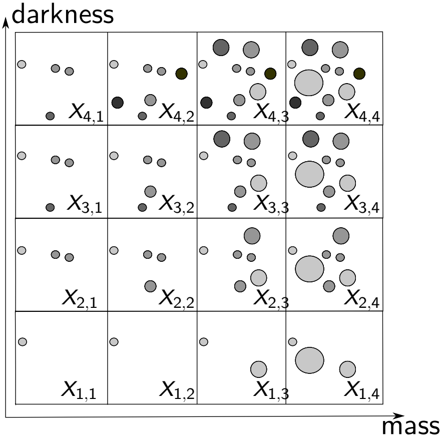
1-D vs. multi-D Persistence

1-D persistence captures the topology of a one-parameter filtration.



1-D vs. multi-D Persistence

Multi-D persistence captures the topology of a family of spaces filtered along multiple geometric dimensions.



Filtrations

- Sublevelset filtrations: Any continuous function $f = (f_1, \ldots, f_k) : X \to \mathbb{R}^k$ induces sub-level sets:

$$X_\alpha = \bigcap_{i=1}^{k} f_i^{-1}((-\infty, \alpha_i]), \qquad \alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k.$$

  Setting

$$\alpha = (\alpha_i) \preceq \beta = (\beta_i) \text{ iff } \alpha_i \leq \beta_i \text{ for every } i$$

  we get a k-parameter filtration of X by sub-level sets:

$$\alpha \preceq \beta \text{ implies } X_\alpha \subseteq X_\beta.$$

- Discrete filtrations: Given a simplicial complex $\mathscr{K}$ and a function $\varphi : \mathscr{V}(K) \to \mathbb{R}^k$, for any $\alpha \in \mathbb{R}^k$ let

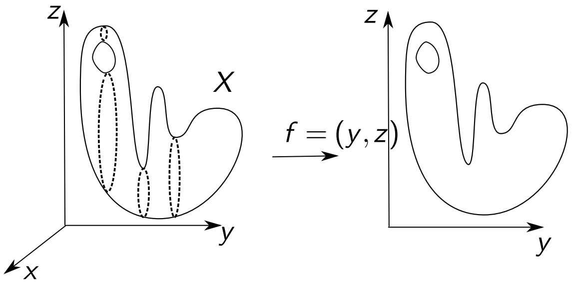
$$\mathscr{K}_\alpha = \{\sigma \in \mathscr{K} \,|\, \varphi(v) \preceq \alpha \text{ for all vertices } v \leq \sigma\}.$$

Rank invariant

For a filtration $\mathscr{F} = \{X_\alpha\}_{\alpha \in \mathbb{R}^k}$ on a triangulable subspace of some $\mathbb{R}^d$,

$$\rho_{\mathscr{F}} : \{(\alpha,\beta) \in \mathbb{R}^k \times \mathbb{R}^k | \alpha \prec \beta\} \to \mathbb{N},$$

$$\rho_{\mathscr{F}}(\alpha,\beta) = \dim \mathrm{im}\, H_*(X_\alpha \hookrightarrow X_\beta).$$



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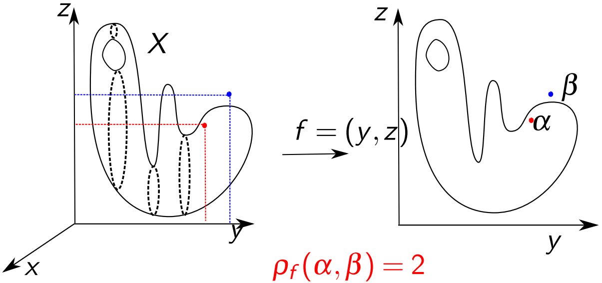

$\rho_f(\alpha, \beta) = 2$

Rank invariant

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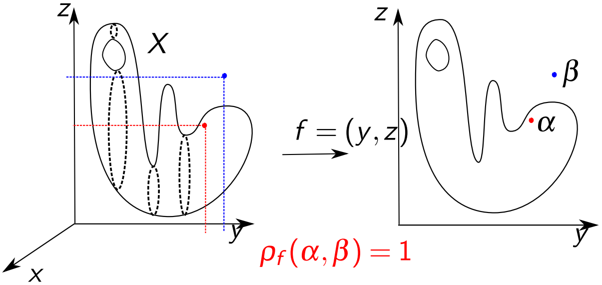

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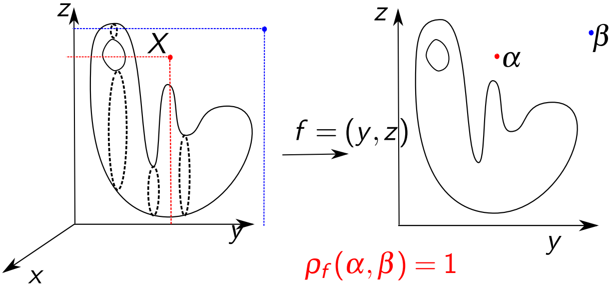

$\rho_f(\alpha, \beta) = 1$

Continuous vs discrete setting

- Sub-level set filtrations are those for which stability results hold:
  $\forall f, f' : X \to \mathbb{R}^k$ continuous functions, $D(\rho_f, \rho_{f'}) \leq \|f - f'\|_\infty$.

Continuous vs discrete setting

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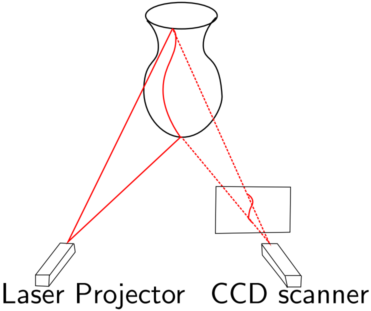
- Discrete filtrations are those actually used in computations:



Stable comparison of rank invariants obtained from discrete data?

From discrete to continuous filtrations

Question: How to extend $\varphi : \mathscr{V}(K) \to \mathbb{R}^k$ to a continuous function $K \to \mathbb{R}^k$ so that its sub-level set filtration coincides with $\{K_\alpha\}_{\alpha \in \mathbb{R}^k}$?

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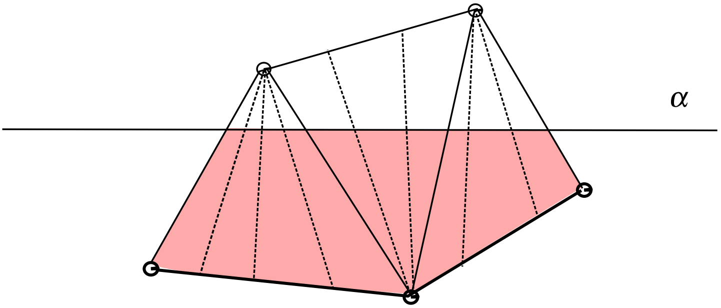
Answer: 1-D persistence: use linear interpolation [Morozov, 2008]



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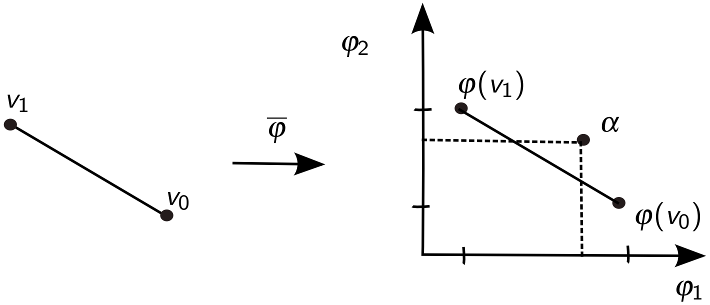
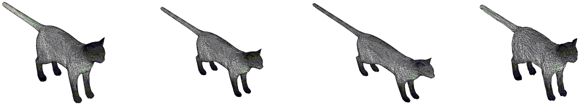
Answer: Multi-D persistence:

linear interpolation yields topological aliasing



Topological Aliasing: numerical experiments



	Original	Linear int.	% Diff
	cat0 vs. cat0-tran1-1		
H_1	0.046150	0.040576	**-13.737185**
H_0	0.225394	0.207266	**-8.746249**
	cat0-tran1-2 vs. cat0-tran2-1		
H_1	0.034314	0.029188	**-17.562012**
H_0	0.208451	0.204511	**-1.926547**
	cat0-tran2-1 vs. cat0-tran2-2		
H_1	0.045545	0.037061	**-22.891989**
H_0	0.212733	0.208097	**-2.227807**

Axis-wise linear interpolation

- Given any $\sigma \in \mathscr{K}$, set $\mu(\sigma) = max\{\varphi(v) \,|\, v$ is a vertex of $\sigma\}$.
- Use induction to define $\varphi^{\urcorner} : K \to \mathbb{R}^k$ on σ and a point $w_\sigma \in \sigma$ s.t.
  - For all $x \in \sigma$, $\varphi^{\urcorner}(x) \preceq \varphi^{\urcorner}(w_\sigma) = \mu(\sigma)$;
  - $\varphi^{\urcorner}$ is linear on any line segment $[w_\sigma, y]$ with $y \in \partial\sigma$.

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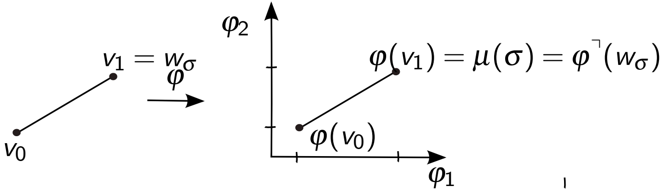

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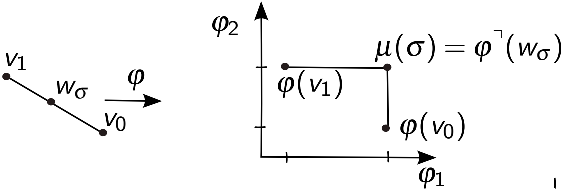

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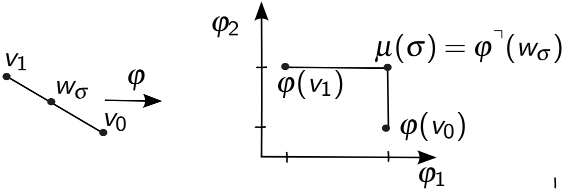

Theorem

For any $\alpha \in \mathbb{R}^k$, K_α is a strong deformation retract of $K_{\varphi^{\urcorner} \preceq \alpha}$.

Bridging stability from continuous to discrete persistence

- X and Y homeomorphic triangulable spaces (real objects);
- $f : X \to \mathbb{R}^k$,$g : Y \to \mathbb{R}^k$ continuous functions (real measurements);
- $\mathscr{K}'$ and $\mathscr{L}'$ simplicial complexes with $|\mathscr{K}'| = K$, $|\mathscr{K}'| = L$ (approximated object);
- $\tilde{\varphi} : K \to \mathbb{R}^k$, $\tilde{\psi} : L \to \mathbb{R}^k$ continuous functions (approximated measurements);

Theorem: If two homeomorphisms $\xi : K \to X$, $\zeta : L \to Y$ exist s.t.

$$\|\tilde{\varphi} - f \circ \xi\|_\infty \leq \varepsilon/4, \quad \|\tilde{\psi} - g \circ \zeta\|_\infty \leq \varepsilon/4$$

then, for any sufficiently fine subdivision $\mathscr{K}$ of $\mathscr{K}'$ and $\mathscr{L}$ of $\mathscr{L}'$,

$$\left|\mathrm{D}(\rho_f, \rho_g) - \mathrm{D}(\rho_\varphi, \rho_\psi)\right| \leq \varepsilon,$$

$\varphi : \mathscr{V}(\mathscr{K}) \to \mathbb{R}^k$, $\psi : \mathscr{V}(\mathscr{L}) \to \mathbb{R}^k$ being restrictions of $\tilde{\varphi}$ and $\tilde{\psi}$.

Sketch of the proof

- $\exists\delta > 0$ s.t. $\max\{\operatorname{diam}\sigma \mid \sigma \in \mathscr{K}$ or $\sigma \in \mathscr{L}\} < \delta \implies$

$$|\mathrm{D}(\rho_{\tilde{\varphi}}, \rho_{\tilde{\psi}}) - \mathrm{D}(\rho_{\varphi^\neg}, \rho_{\psi^\neg})| < \varepsilon/2.$$

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$$|\mathrm{D}(\rho_{\tilde{\varphi}}, \rho_{\tilde{\psi}}) - \mathrm{D}(\rho_{\varphi}, \rho_{\psi})| < \varepsilon/2.$$
- $$\begin{aligned} \mathrm{D}(\rho_f, \rho_g) &\leq \mathrm{D}(\rho_f, \rho_{f\circ\xi}) + \mathrm{D}(\rho_{f\circ\xi}, \rho_{\tilde{\varphi}}) + \mathrm{D}(\rho_{\tilde{\varphi}}, \rho_{\tilde{\psi}}) \\ &+ \mathrm{D}(\rho_{\tilde{\psi}}, \rho_{g\circ\zeta}) + \mathrm{D}(\rho_{g\circ\zeta}, \rho_g) \end{aligned}$$
  .

$\square$

Applications to model precision concerns

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For a dataset of 5000 functions $f_i : T \to \mathbb{R}^2$ on the torus T, given a set of triangulations of T with 2^{2N} simplices (varying N) we obtain the function $\varphi_{i,N}$ by sampling f_i at the vertices of the triangulations.

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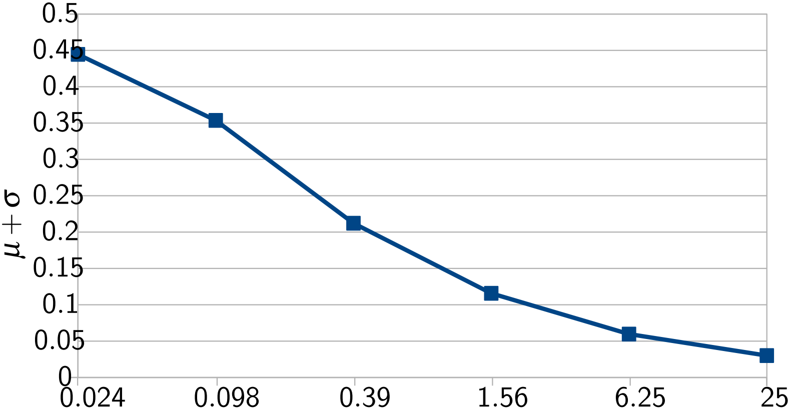
We can estimate the error caused by coarsening the model by computing $\|\varphi_{i,N} - f_i\|_\infty$:

N	4	5	6	7	8	9
μ	0.3841	0.2995	0.1785	0.0977	0.0503	0.0254
σ	0.060	0.0541	0.0335	0.0179	0.0092	0.0046
$\mu + \sigma$	0.4444	0.3536	0.2120	0.1157	0.0596	0.0300

Applications to model precision concerns

By the Stability Theorem we get a bound of the error on the rank invariants caused by model coarsening



Conclusions

We have shown that in multidimensional persistence:

- Passing from discrete to continuous setting, a peculiar phenomenon occurs: topological aliasing
- Topological aliasing is removed by using axis-wise linear interpolation
- Stability of rank invariants passes from continuous to discrete filtrations
- Stability for discrete filtrations yields a method for bounding the error caused by model coarsening

THANK YOU FOR YOUR ATTENTION!