

Homotopy theories of dynamical systems

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Dynamical systems

A *dynamical system* (or *S-dynamical system*, or *S-space*) is a map of simplicial sets

$$\phi : X \times S \rightarrow X,$$

giving an action of a *parameter space* S on a *state space* X .

Equivalently, a dynamical system is a map

$$\phi_* : S \rightarrow \mathbf{hom}(X, X)$$

into the topological monoid of endomorphisms of X .

$s \mapsto \phi_*(s) : X \rightarrow X$ is continuous in $s \in S$.

If S has a monoidal structure, then ϕ_* is required to be a homomorphism.

Most often, X is a manifold, and S is a time parameter which is a submanifold of the real numbers.

Examples: Discrete dynamical systems

If $S = *$ is a one-point space, then a dynamical system parameterized by S is just a map $X \rightarrow X$.

The free monoid on the one-point space is a copy of \mathbb{N} , and so there is an associated monoid map

$$f_* : \mathbb{N} \rightarrow \mathbf{hom}(X, X)$$

Cellular automata: $X = (\mathbb{Z}^n)^k$ consists of points in an integral lattice, each of which can be in a set of k states.

Category of S -spaces

A *morphism* $f : X \rightarrow Y$ of S -spaces is a map $f : X \rightarrow Y$ which preserves the respective S -actions. Morphisms are also called *S -equivariant maps*.

S – **sSet** is the category of S -spaces and their morphisms.

Question: (Carlsson) What could be meant by a homotopy theory of dynamical systems, or S -spaces?

Naive Definition: A map $X \rightarrow Y$ of S -spaces is a *weak equivalence* if and only if the underlying map of simplicial sets (spaces) is a weak equivalence.

This is analogous to the traditional naive definition of G -equivariant weak equivalence for spaces equipped with an action by a group G .

Varying the parameter space

It should mean something in the homotopy theory of dynamical systems if the parameter space S is contractible.

We need a category of dynamical systems which contains the S -space categories for all parameter spaces S , and for which we can vary S .

A map $(\theta, f) : X \rightarrow Y$ consists of maps $\theta : S \rightarrow T$ and $f : X \rightarrow Y$ such that the following commutes:

$$\begin{array}{ccc} S \times X & \rightarrow & X \\ \theta \times f \downarrow & & \downarrow f \\ T \times Y & \rightarrow & Y \end{array}$$

There is a homotopy theory for this category, but the weak equivalences are more difficult to describe.

Feel good fact: if θ and f are weak equivalences, then (θ, f) is a weak equivalence for this theory (whatever it is).

Quillen model categories

A *closed model category* is a category \mathcal{M} equipped with *weak equivalences*, *fibrations* and *cofibrations* s.t. the following hold:

CM1: \mathcal{M} has all limits and colimits.

CM2: If any two of f , g , $g \cdot f$ is a weak equivalence, so is the third.

CM3: Weak equivalences, cofibrations and fibrations are closed under retraction.

CM4: Given a cofibration i , a fibration p and diagram

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

then the lift exists if either i or p is a weak equivalence (trivial).

CM5: Every f has $f = p \cdot j = q \cdot i$, where p is a fibration, j is a triv. cofibration, q is a triv. fibration, i is a cofibration.

Examples: ordinary homotopy theory

$s\mathbf{Set}$ = simplicial sets, and \mathbf{Top} = topological spaces.

Fibrations for \mathbf{Top} are Serre fibrations, and weak equivalences are weak homotopy equivalences. CW -complexes are cofibrant objects.

There are adjoint functors

$$| \cdot | : s\mathbf{Set} \rightleftarrows \mathbf{Top} : S$$

The weak equivalences $X \rightarrow Y$ of $s\mathbf{Set}$ are those maps which induce weak equivalences $|X| \rightarrow |Y|$, and the cofibrations are monomorphisms. Fibrations are Kan fibrations.

The adjoint functors form a “Quillen equivalence”, and induce an adjoint equivalence of homotopy categories

$$| \cdot | : \mathrm{Ho}(s\mathbf{Set}) \rightleftarrows \mathrm{Ho}(\mathbf{Top}) : S$$

Homotopy theory of S -spaces, 1

A map $f : X \rightarrow Y$ of S -spaces is a

- 1) *weak equivalence* if f is a weak equivalence of simplicial sets
- 2) *cofibration* if f is a monomorphism
- 3) *projective fibration* if f is a Kan fibration.

An *injective fibration* is a map which has the right lifting property (RLP) with respect to all trivial cofibrations.

A *projective cofibration* is a map which has the left lifting property (LLP) with respect to all trivial projective fibrations.

$$\begin{array}{ccc} A & \longrightarrow & X \\ i \downarrow & \nearrow & \downarrow p \\ B & \longrightarrow & Y \end{array}$$

Theorem

Suppose that S is a fixed choice of parameter space.

- 1) The category $S - \mathbf{sSet}$, together with the cofibrations, weak equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*
- 2) The category $S - \mathbf{sSet}$, together with the projective cofibrations, weak equivalences and projective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

The proof follows a pattern that we know: p is an injective fibration if and only if it has the RLP wrt all bounded trivial cofibrations, and part 1) implies part 2).

Dynamical systems to diagrams

$F(S)$ is the free simplicial monoid associated to a space S :

$$F(S) = * \sqcup S \sqcup S^{\times 2} \sqcup S^{\times 3} \sqcup \dots$$

and an S -space $X \times S \rightarrow X$ is canonically a module over $F(S)$.

Alternatively, $F(S)$ is a simplicial category (or a category enriched in simplicial sets, with one object) and X is an $F(S)$ -*diagram*.

Definition: A *simplicial category* A is a simplicial object in categories.

A consists of simplicial sets $\text{Ob}(A)$ and $\text{Mor}(A)$ such that all categorical structure $s, t : \text{Mor}(A) \rightarrow \text{Ob}(A)$, $e : \text{Ob}(A) \rightarrow \text{Mor}(A)$, compositions, are compatible with the simplicial structure.

Definition: A *category enriched in simplicial sets* is a simplicial category B such that $\text{Ob}(B)$ is discrete (ie. generated by vertices).

Internal diagrams

A = simplicial category. An A -diagram in simplicial sets consists of a simplicial set map $\pi : X \rightarrow \text{Ob}(A)$ and an action diagram

$$\begin{array}{ccc} X \times_s \text{Mor}(A) & \xrightarrow{m} & X & (x, \alpha) \mapsto \alpha(x) \\ \downarrow & & \downarrow \pi & \\ \text{Mor}(A) & \xrightarrow{t} & \text{Ob}(A) & \end{array}$$

such that $1(x) = x$ and $\beta(\alpha(x)) = (\beta\alpha)(x)$.

\mathbf{Set}^A is the category of A -diagrams. A *morphism* (natural transformation) is a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \pi \searrow & & \swarrow \pi \\ & \text{Ob}(A) & \end{array}$$

which respects the multiplication.

Example: Ordinary functors

A functor $F : I \rightarrow \mathbf{Set}$ consists of sets $F(i)$, $i \in \text{Ob}(I)$, and morphisms $F(\alpha) : F(i) \rightarrow F(j)$ satisfying the usual properties.

Alternatively, F consists of a function

$$\pi : F = \bigsqcup_{i \in \text{Ob}(I)} F(i) \rightarrow \bigsqcup_{i \in \text{Ob}(I)} * = \text{Ob}(I),$$

and a morphism

$$m : F \times_s \text{Mor}(I) = \bigsqcup_{\alpha: i \rightarrow j} F(i) \rightarrow \bigsqcup_j F(j) = F$$

A natural transformation of functors $\alpha : F \rightarrow G$ is a function

$$\bigsqcup_{i \in \text{Ob}(I)} F(i) \rightarrow \bigsqcup_{i \in \text{Ob}(I)} G(i)$$

which is fibred over $\text{Ob}(I)$.

Homotopy theory of diagrams, 1

A = category enriched in simplicial sets (ie. $\text{Ob}(A)$ is discrete).

A map $f : X \rightarrow Y$ of A -diagrams is

- 1) a *weak equivalence* if the map

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow & \swarrow \\ & \text{Ob}(A) & \end{array}$$

is a weak equivalence of $s\mathbf{Set}/\text{Ob}(A)$

- 2) a *cofibration* if the simplicial set map f is a monomorphism
- 3) a *projective fibration* if the simplicial set map f is a Kan fibration.

An *injective fibration* is a map which has the right lifting property with respect to all trivial cofibrations.

A *projective cofibration* is a map which has the left lifting property with respect to all trivial projective fibrations.

Theorem

Suppose that A is a category which is enriched in simplicial sets.

- 1) The category \mathbf{Set}^A , together with the cofibrations, weak equivalences and injective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*
- 2) The category \mathbf{Set}^A , together with the projective cofibrations, weak equivalences and projective fibrations, satisfies the axioms for a proper closed simplicial model category. This model structure is cofibrantly generated.*

The theorem is a special case of a result which holds for diagrams of simplicial presheaves over a presheaf of simplicial categories with discrete objects.

Homotopy colimits

Suppose that $F : I \rightarrow \mathbf{Set}$ is an ordinary functor.

There is a category $E_I F$ whose objects are the pairs (x, i) with $x \in F(i)$. The morphisms $\alpha : (x, i) \rightarrow (y, j)$ are morphisms $\alpha : i \rightarrow j$ of I such that $\alpha_*(x) = y$.

This category has a nerve $B(E_I F)$, whose n -simplices are strings

$$(x_0, i_0) \xrightarrow{\alpha_1} (x_1, i_1) \xrightarrow{\alpha_2} \dots \xrightarrow{\alpha_n} (x_n, i_n)$$

of length n . All that matters here is x_0 and the string in I :

$$\underline{\mathrm{holim}}_I F_n = B(E_I F)_n = \bigsqcup_{i_0 \rightarrow \dots \rightarrow i_n} F(i_0).$$

This is the *homotopy colimit* for the functor F . It is the space of *finite trajectories* associated to the functor F , or the *space of dynamics* for the functor F .

Homotopy colimits and pullbacks

$A =$ simplicial category. Every A -diagram $\pi : X \rightarrow \text{Ob}(A)$ determines a bisimplicial set map

$$\underline{\text{holim}}_A X \rightarrow BA,$$

by taking homotopy colimit in each simp. degree, giving a functor

$$\underline{\text{holim}}_A : \mathbf{Set}^A \rightarrow s^2\mathbf{Set}/BA.$$

The pullback functor

$$\text{pb} : s^2\mathbf{Set}/BA \rightarrow \mathbf{Set}^A,$$

is defined by taking diagonals of the pullbacks

$$\begin{array}{ccc} \text{pb}(Y)_i & \longrightarrow & Y \\ \downarrow & & \downarrow \\ B(A/i) & \longrightarrow & BA \end{array}$$

in all simplicial degrees.

Homotopy colimit structure

A map $f : X \rightarrow Y$ of A -diagrams is a *cofibration* if the underlying simplicial set map is a monomorphism.

$f : X \rightarrow Y$ is a *weak equivalence* if the induced map

$$\underline{\operatorname{holim}}_A X \rightarrow \underline{\operatorname{holim}}_A Y$$

is a *diagonal weak equivalence* of bisimplicial sets.

Theorem

With these definitions, the category \mathbf{Set}^A satisfies the axioms for a proper closed model category.

This is the *homotopy colimit model structure* for the category of A -diagrams. Schlichtkrull uses a special case in his proof of the Barratt-Kahn-Priddy-Quillen Theorem ($QS_0^0 \simeq (B\Sigma_\infty)_+$).

Relation with diagonal model structure

Theorem

The functors $\underline{\text{holim}}_A$ and pb induce an equivalence of categories

$$\text{Ho}(\mathbf{Set}^A) \simeq \text{Ho}(s^2\mathbf{Set}/BA)$$

for the homotopy colimit structure on \mathbf{Set}^A and the diagonal structure on $s^2\mathbf{Set}/BA$.

A bisimplicial set map $f : X \rightarrow Y$ is a *cofibration* if it is a monomorphism. $f : X \rightarrow Y$ is a *diagonal weak equivalence* if the simplicial set map $f_* : d(X) \rightarrow d(Y)$ is a weak equivalence.

Theorem

There is a model structure on the category $s^2\mathbf{Set}$ for which the cofibrations are the monomorphisms and the weak equivalences are those map $X \rightarrow Y$ which induce a weak equivalence $d(X) \rightarrow d(Y)$ of associated diagonal simplicial sets.



Homotopy types of categories

A functor $f : C \rightarrow D$ is a fibration (respectively weak equivalence) if the induced map $BC \rightarrow BD$ is an sd^2 -fibration (respectively weak equivalence).

Theorem (Thomason)

- 1) *With these definitions the category **Cat** of small categories has the structure of a proper closed model category.*
- 2) *The adjunction*

$$P : s\mathbf{Set} \rightleftarrows \mathbf{Cat} : B$$

is a Quillen equivalence, for the sd^2 -structure on simplicial sets.

P is the *path category* functor.

$PBC \cong C$ for small categories C .

Homotopy types of categories, 2

Crux of the proof: if $K \rightarrow L$ is an inclusion of finite simplicial complexes, then all pushout diagrams

$$\begin{array}{ccc} NBN(K) & \rightarrow & C \\ \downarrow & & \downarrow \\ NBN(L) & \rightarrow & D \end{array}$$

induce homotopy cocartesian diagrams of simplicial sets.

$N(K)$ is the poset of non-degenerate simplices of a simplicial set K : $\sigma \leq \tau$ if $\sigma \in \langle \tau \rangle$.

$BN(K) = \text{sd}(K)$ (*order complex* of NK) is the *barycentric subdivision* of K if K is a simplicial complex.

Note that $BNBN(K) \cong \text{sd}^2(K)$ for simplicial complexes K .

The induced functor $NBN(K) \rightarrow NBN(L)$ is a “Dwyer map”.

Subdivisions

The *subdivision*

$$\mathrm{sd}(X) = \lim_{\Delta^n \rightarrow X} \mathrm{BN}\Delta^n.$$

is a colimit of barycentric subdivisions of simplices.

$$\mathrm{Ex}(X)_n = \mathrm{hom}(\mathrm{sd}(\Delta^n), X)$$

for simplicial sets X .

There are adjoint functors

$$\mathrm{sd}^n : \mathbf{sSet} \rightleftarrows \mathbf{sSet} : \mathrm{Ex}^n$$

and natural weak equivalences $\mathrm{sd}^n X \xrightarrow{\simeq} X$ and $Y \xrightarrow{\simeq} \mathrm{Ex}^n Y$ for all simplicial sets X and Y .

Subdivision model structures

$p : X \rightarrow Y$ is an sd^n -fibration if the map $Ex^n X \rightarrow Ex^n Y$ is a Kan fibration, or if p has the RLP wrt all $sd^n(\Lambda_k^m) \rightarrow sd^n(\Delta^m)$.

The sd^n -cofibrations are those maps which have the LLP w.r.t. all maps which are sd^n -fibrations and weak equivalences.

Theorem

- 1) The category $s\mathbf{Set}$ of simplicial sets, together with the weak equivalences, sd^n -fibrations, and sd^n -cofibrations, satisfies the axioms for a proper closed model category.
- 2) The adjoint pair of functors

$$sd^n : s\mathbf{Set} \rightleftarrows s\mathbf{Set} : Ex^n$$

defines a Quillen equivalence between the standard model structure and the sd^n -structure for simplicial sets.

Subdivisions for bisimplicial sets

$$\mathrm{sd}^{m,n} X = \varinjlim_{\Delta^{p,q} \rightarrow X} \mathrm{sd}^m \Delta^p \tilde{\times} \mathrm{sd}^n \Delta^q.$$

$\tilde{\times}$ is external product: $\Delta^{p,q} = \Delta^p \tilde{\times} \Delta^q$.

The functor $\mathrm{sd}^{m,n}$ has a right adjoint $\mathrm{Ex}^{m,n}$. Both functors preserve diagonal homotopy types.

A map $f : X \rightarrow Y$ of bisimplicial sets is an *$\mathrm{sd}^{m,n}$ -fibration* if the induced map $\mathrm{Ex}^{m,n} X \rightarrow \mathrm{Ex}^{m,n} Y$ is a diagonal fibration.

$\mathrm{sd}^{m,n}$ -cofibrations are defined by a left lifting property with respect to trivial fibrations.

Theorem

- 1) *The category $s^2\mathbf{Set}$, with the $sd^{m,n}$ -fibrations, diagonal weak equivalences and $sd^{m,n}$ -cofibrations, satisfies the axioms for a proper closed model category.*
- 2) *The adjoint functors*

$$sd^{m,n} : s^2\mathbf{Set} \rightleftarrows s^2\mathbf{Set} : Ex^{m,n}$$

define a Quillen equivalence between the diagonal model structure and the $sd^{m,n}$ -structure for bisimplicial sets.

Homotopy types of simplicial categories

A morphism $f : C \rightarrow D$ of simplicial categories is

- a) a *fibration* if the map $BC \rightarrow BD$ is an $sd^{2,0}$ -fibration, and
- b) a *weak equivalence* if the map $BC \rightarrow BD$ is a diagonal equivalence.

Theorem

- 1) *With these definitions, the category $s\mathbf{Cat}$ satisfies the axioms for a proper closed model category.*
- 2) *The adjunction*

$$P : s^2\mathbf{Set} \rightleftarrows s\mathbf{Cat} : B$$

defines a Quillen equivalence between simplicial categories and the $sd^{2,0}$ -model structure for bisimplicial sets.

Diagram homotopy types

Let $s\mathbf{Dia}$ be the category whose objects are bisimplicial set maps $X \rightarrow BC$ where C is a simplicial category. Say that a morphism

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ BC & \xrightarrow{g} & BD \end{array}$$

is a weak equivalence if f and g are weak equivalences, and is a fibration if the maps $g : BC \rightarrow BD$ and

$$X \rightarrow BC \times_{BD} Y$$

are $sd^{2,0}$ -fibrations. Say that the map is a cofibration if f is an $sd^{2,0}$ -cofibration and g is a cofibration of simplicial categories.

Theorem

With these definitions, the category $s\mathbf{Dia}$ satisfies the axioms for a closed model category.

- 1) We regard bisimplicial set maps $Y \rightarrow BA$ as A -diagrams, but that's okay: every A -diagram Y can be recovered from the map $\text{holim}_{\rightarrow A} Y \rightarrow BA$ up to sectionwise weak equivalence, via the pullback functor.
- 2) We now have a homotopy theory for *all* dynamical systems: dynamical systems and parameter spaces can be varied simultaneously.
- 3) All of this can be topologized. There is a model structure for presheaves of simplicial categories which is defined by an $\text{sd}^{2,0}$ -model structure for bisimplicial presheaves, and a corresponding model structure on diagram objects $Y \rightarrow BA$ in bisimplicial presheaves, all over an arbitrary Grothendieck site.

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