On the geometric nature of mutual exclusion

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Work in progress

CEA LIST and Ecole Polytechnique, France

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Contents of the talk

Mostly trying to wrap things up...

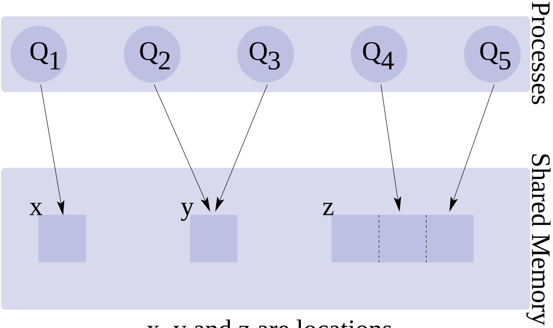
Motivation: study of concurrent systems and their schedules; talk influenced by:

- The surprising efficiency of classical mutual exclusion models in the analysis of concurrent systems
- Geometric group theory, and the ATMCS 2004 talk by R. Ghrist, work on trace spaces by M. Raussen, universal dicoverings of L. Fajstrup, our paper in GETCO 2010 etc.

In the particular case of mutual exclusion

- Corresponds exactly to the NPC cubical complexes
- CAT(0) cubical complexes are prime event structures
- Classical unfoldings of prime event structures/safe Petri nets are universal dicoverings of NPC cubical complexes
- As a potential application: "directed" H_1 in the NPC case

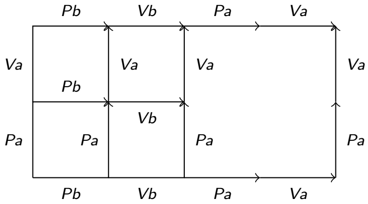
Original motivation of this work



x, y and z are locations

Not sequential programs, bad states, chaotic behavior
$\Longrightarrow$ Need for synchronizations $\Longrightarrow$ Need for locks: Py, Vy (binary and counting semaphores
$\Longrightarrow$ Interleaving semantics given by a "shuffle" of transition systems (or fibred product) - Can we "do" better?

$$P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$$



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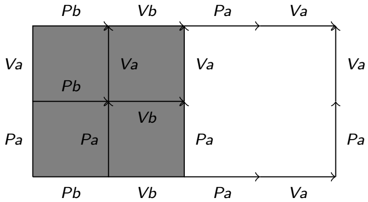

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Models

On the "geometry" side

- Cubical sets (pre-existing the field of course!)
- Po-spaces (i.e. topological space with closed partial order), introduced first in other fields (domain theory P. Johnstone etc., functionnal analysis L. Nachbin etc.) local po-spaces (atlas of po-spaces - L. Fajstrup, E. Goubault, M. Raussen)
- d-spaces (M. Grandis)
- Flows (P. Gaucher)
- Streams (S. Krishnan)
- etc.

More classical models in computer science

Transition systems, prime event structures, Petri nets etc.

ÉCOLE POLYTECHNIQUE

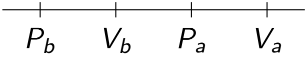
Geometric semantics

A program

$$\color{red}{P_b}\texttt{;x:=1;}\color{red}{V_b}\texttt{;}\color{red}{P_a}\texttt{;y:=2;}\color{red}{V_a} \mid \color{red}{P_a}\texttt{;y:=3;}\color{red}{V_a}$$

will be interpreted as a **directed space**:

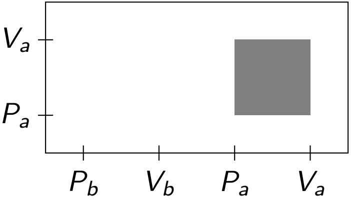
- $P_b.V_b.P_a.V_a$



- $P_a.V_a$

- $P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$

Forbidden regions



A program

$$P_b\,;\texttt{x:=1};\,V_b\,;\,P_a\,;\texttt{y:=2};\,V_a \mid P_a\,;\texttt{y:=3};\,V_a$$

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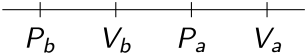

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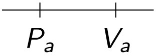

- $P_b.V_b.P_a.V_a \quad | \quad P_a.V_a$

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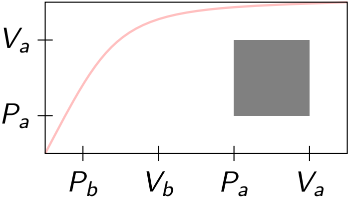

A program

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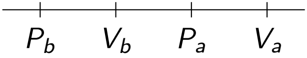

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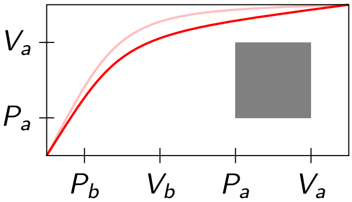


Directed homotopies

A directed homotopy (with fixed extremities) between directed paths f and g $\phi : f \to g : \overrightarrow{I} \to Y$ is a directed map $\phi : X \times \overrightarrow{I} \to Y$ such that:

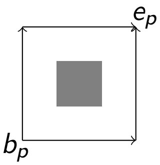
- for all $x \in [0,1]$, $\phi(x,0) = f(x)$, $\phi(x,1) = g(x)$
- for all $t \in [0,1]$, $\phi(0,t) = f(0)$, $\phi(1,t) = f(1)$

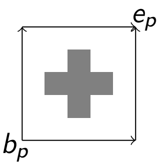
(in particular, for all $t \in [0,1]$, $\phi(.,t)$ is a directed path)

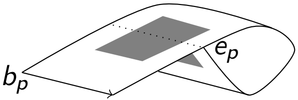
Schedules are directed paths up to directed homotopies. How to compute them? (fundamental category $\overrightarrow{\pi}_1(X)$, category of components, trace spaces... still quite computationally demanding!)

To each program p we associate a d-space (H_p, b_p, e_p):

$P_a.V_a|P_a.V_a$ $\quad$ $P_a.P_b.V_b.V_a|P_b.P_a.V_a.V_b$ $\quad$ $P_a.(V_a.P_a)^*|P_a.V_a$

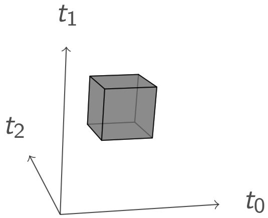




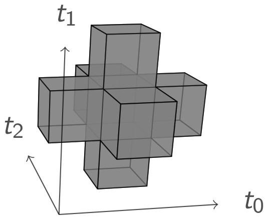


Examples of geometric semantics

$$P_a.V_a|P_a.V_a|P_a.V_a$$
$$(\kappa_a = 2)$$

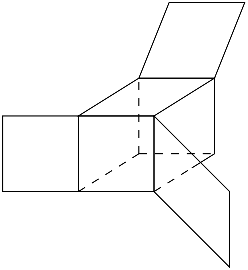


$$P_a.V_a|P_a.V_a|P_a.V_a$$
$$(\kappa_a = 1)$$



Geometric realization

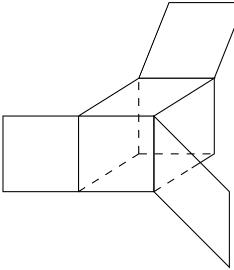
Maps pre-cubical sets to a gluing of hyper-cubes $[0,1]^n$ along their faces (with quotient topology)



(directed path structure agreeing with partial order on each hypercube)

Produces cubical complexes

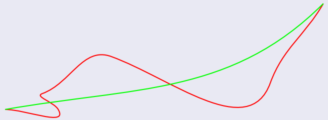
- Topological spaces X that are particular cellular complexes whose building blocks (cells) are n-cubes, i.e. $[0,1]^n$
- In particular we have, as data, maps $c_i^n : [0,1]^n \to X$ identifying the ith cell of dimension n in X (homeomorphism from the interior of $[0,1]^n$ onto X plus some extra properties for the boundaries)



(directed path structure agreeing with partial order on each hypercube)

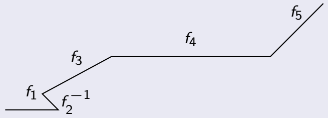
Adding geodesic metric structure

Idea: identify directed paths with shortest paths...



Undirected path, "longer length" than the directed one

Or on π_1...



$f_1 f_2^{-1} f_3 f_4 f_5$ is a "longer word" in π_1 (seen as generated by a semi-group of "directed" generators)

When can we give a meaning to this?

d_2 and d_∞ metrics on a (arc connected) cubical complex X

- Each of the cell $c_i^n((0,1)^n)$ inherits from the d_k (here $k = 2$ or ∞) metrics by:

$$d_k(x, y) = d_k\left((c_i^n)^{-1}(x), (c_i^n)^{-1}(y)\right)$$

- For any two points x, y in X,

$$d_k(x, y) = \inf_{\gamma \in \Gamma} \sum_{i=0,n_\gamma-1} d_k\left(\gamma(t_k), \gamma(t_{k+1})\right)$$

  where Γ is the set of paths from x to y and $(\gamma(t_k), \gamma(t_{k+1}))$ belongs to the same k-cell of X (hence $d_k(\gamma(t_k), \gamma(t_{k+1}))$ is well-defined)

Arc-length

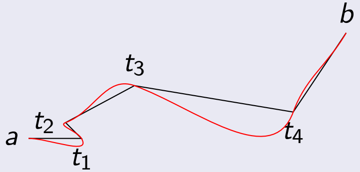
Let $c : [a, b] \to X$ a curve in X, define its arc-length as:

$$l(c) = \sup_{a=t_0 \leqslant t_1 \leqslant \ldots \leqslant t_n = b} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1}))$$

(the supremum is taken over all n and all possible partitions)

Rectifiability

All paths in a finitely generated d_∞-space have finite length.



Local geodesic

A map $c : [a, b] \rightarrow X$ is a local geodesic is for all $t \in [a, b]$, there exists $\epsilon > 0$ such that

$$d(c(t'), c(t'')) = \mid t' - t'' \mid$$

for all t', $t'' \in [a, b]$ with $\mid t - t' \mid + \mid t - t'' \mid \leqslant \epsilon$

Geodesic

A map $c : [0, l] \rightarrow X$ is a geodesic (from $c(a)$ to $c(b)$) if

$$d(c(t), c(t')) = \mid t - t' \mid$$

for all $t, t' \in [0, l]$ (in particular $l = l(c) = d(x, y)$).

Geodesic metric space

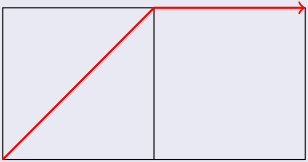
- Such metrics give connected cubical complexes the structure of a geodesic metric space, i.e. any two points can be linked by a geodesic
- (complete in case it is a finite cubical complexes - Bridson & Haefliger 1999)

Extension to more general d-spaces?

Open question...

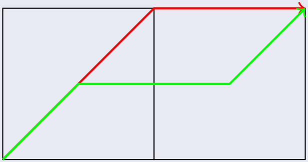
Example: 2 unit 2-cells

"Left-greedy" [Ghrist] - arc length 2



EXAMPLE: 2 UNIT 2-CELLS

More of them (still arc length 2 for d_∞!):



d_∞ geodesics (start to end)

Example: 2 unit 2-cells

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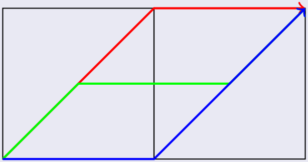

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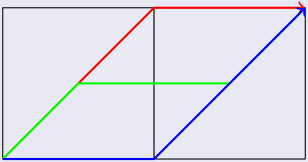

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Example: 2 unit 2-cells

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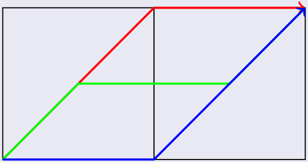
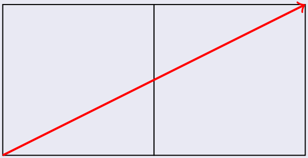


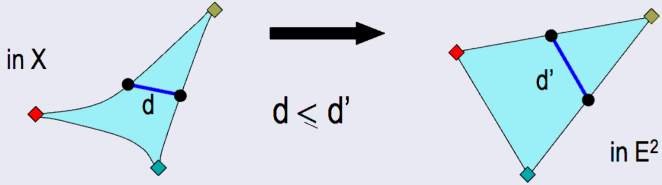
Of course, a unique d_2 geodesic!

(d_2 arc length is $\sqrt{5} > 2$)



CAT(0) and NPC

CAT(0) geometry

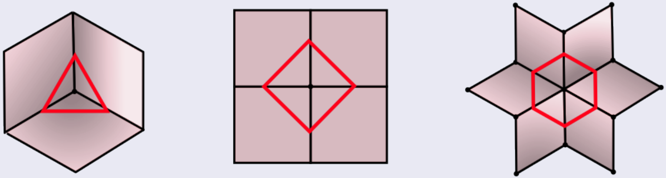


X is CAT(0) iff all triangles are no fatter than in the planar case

X has non-positive curvature (NPC) iff X is locally $CAT(0)$.

Combinatorial condition

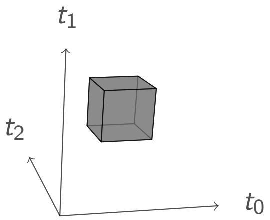
Gromov's condition



- Property: a d_∞-space is NPC iff its link at every vertex is a flag complex ("if the red edges look like a k-simplex, then there is really a k-simplex linking them")
- Implies: empty squares are OK, but not empty cubes nor empty hypercubes of dimension $\geqslant 3$

NPC IS EXACTLY MUTUAL EXCLUSION MODELS

$$P_a.V_a|P_a.V_a|P_a.V_a$$
$$(\kappa_a = 2)$$



not NPC

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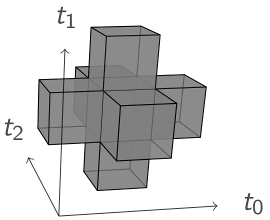
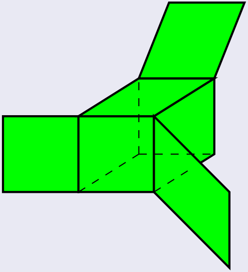


NPC

In the right-hand side case, we know that the trace space (M. Raussen) is discrete, $\overrightarrow{\pi}_1$ is “combinatorial”...can we find this with the geodesic metric approach?

In short

Example

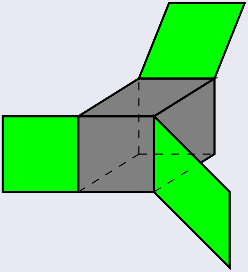


is CAT(0) (the 3-cube is filled-in)

Some facts [Sageev 1997]

- A (globally) CAT(0) cubical set is a "tree of cubes" (contractible space)

Example



is not CAT(0) (the 3-cube is not filled-in)

Some facts [Sageev 1997]

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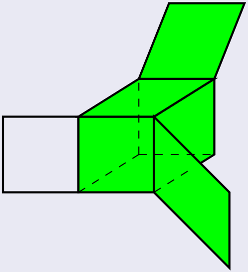
In short

Example



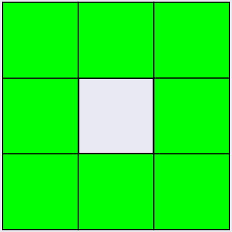
is NPC but not CAT(0) (the 3-cube is filled-in)

Some facts [Sageev 1997]

- A (locally) CAT(0), i.e. NPC cubical set may have some non trivial homotopy

In short

Example



is NPC but not CAT(0) ($PaVa \mid PaVa$)

Some facts [Sageev 1997]

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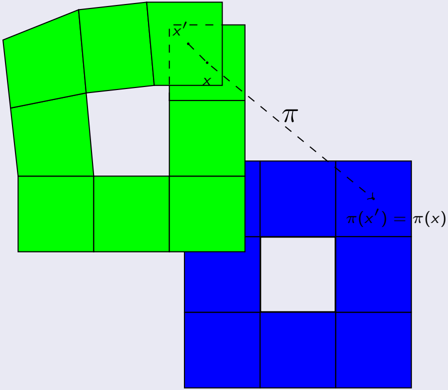
EXAMPLE



Blue is NPC, the space in green is CAT(0)

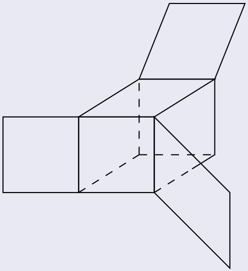
SOME FACTS [SAGEEV 1997]

Some facts [Sageev 1997]

- The universal covering of an NPC cubical set is CAT(0)
- Any graph is NPC
- Any surface of genus $g > 1$ (in particular, a torus) is NPC
- Products of NPC spaces are NPC (the link of the product is the join of the two links)
- (some more to be said on special cube complexes...)

Combinatorial description of CAT(0) cubical complexes

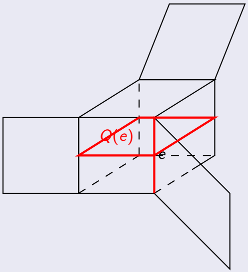
Hyperplanes of a CAT(0) cubical complex



Definitions

- Start with our usual example...

Hyperplanes of a CAT(0) cubical complex

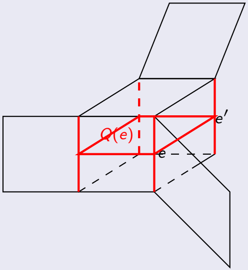


Definitions

- For all n-cubes Q in X, and edges $e \in Q$, $Q(e)$ is the (n-1)-subcube obtained by intersecting Q with the hyperplane orthogonal to e, containing the midpoint of e.

Combinatorial description of CAT(0) cubical complexes

Hyperplanes of a CAT(0) cubical complex

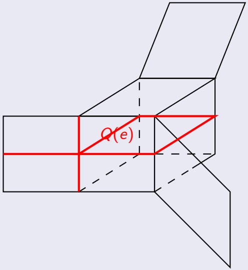


Definitions

- Two edges e, e' are equivalent if $Q(e) = Q(e')$ - plus closure by transitivity.

Combinatorial description of CAT(0) cubical complexes

Hyperplanes of a CAT(0) cubical complex

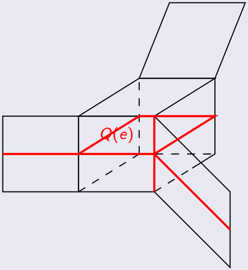


Definitions

- An hyperplane is $H(e) = \cup Q(e)$ for all equivalent edges (to e) and all cubes Q containing e (separates X in exactly two parts)

Combinatorial description of CAT(0) cubical complexes

Hyperplanes of a CAT(0) cubical complex



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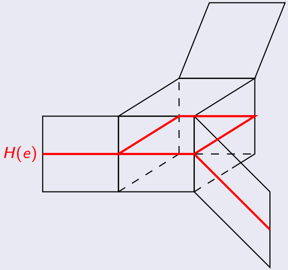

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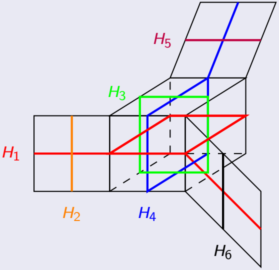
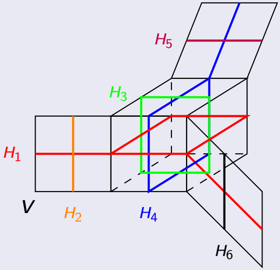


Definitions

- All 6 hyperplanes

CAT(0) (rooted) cubical complexes

Combinatorics of vertices/hyperplanes [Ardila et al. 2011]

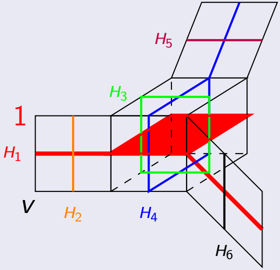


"Recipe"

- Start with the root v - defines the negative half-spaces for each of the H_i

CAT(0) (rooted) cubical complexes

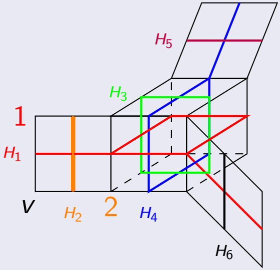
Combinatorics of vertices/hyperplanes [Ardila et al. 2011]



"Recipe"

- Take the unique vertex closest to v on the opposite side of some hyperplane H_i, and iterate

Combinatorics of vertices/hyperplanes [Ardila et al. 2011]



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CAT(0) (rooted) cubical complexes

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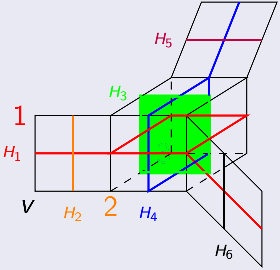

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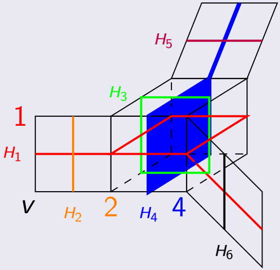

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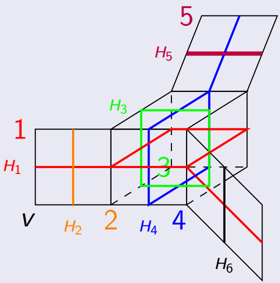

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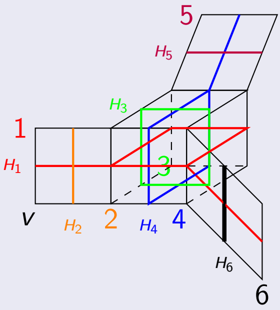

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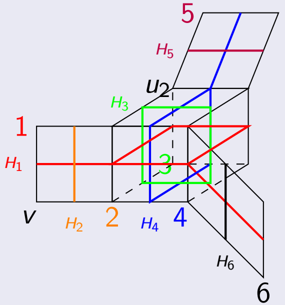
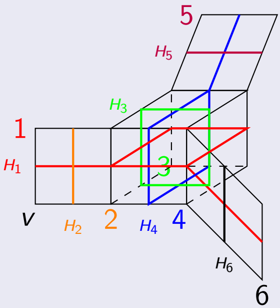


$$
\begin{array}{ccc}
u_1 = v & \leqslant & u_2 \\
\begin{pmatrix} H_1 : - \\ H_2 : - \\ H_3 : - \\ H_4 : - \\ H_5 : - \\ H_6 : - \end{pmatrix} & \leqslant & \begin{pmatrix} + \\ - \\ + \\ - \\ - \\ - \end{pmatrix}
\end{array}
$$

"Recipe"

- Order all vertices: $u_1 < u_2$ if u_1 is in the part of the complex opposite to v with respect to hyperplanes H_j ("j-positive part") implies u_2 as well, for all $j \rightarrow$ poset $L(X, v)$ (X is even a po-space!)

COMBINATORICS OF VERTICES/HYPERPLANES [ARDILA ET AL. 2011]

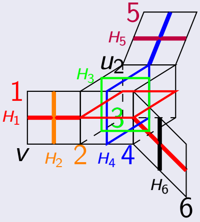


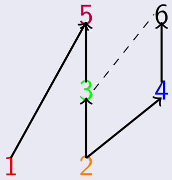
"RECIPE"

- This is a meet-semi lattice - the join-irreducible elements are in bijection with the hyperplanes of the cubical set!

From CAT(0) cubical sets to...event structures

Its poset (with inconsistent pairs) [Ardila et al. 2011]





- Order ($\longrightarrow$) : meet semi-lattice
- Plus conflicts or "inconsistent pairs" (_ _ _) : 3 and 6 are inconsistent since it is impossible to cross <u>once</u> both hyperplanes 6 and 3 from v

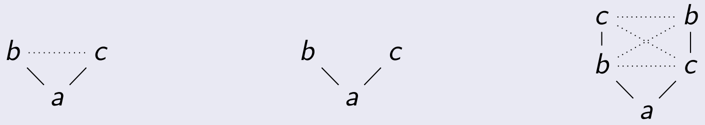
This is nothing else than a prime event structure!

(PRIME) EVENT STRUCTURES

An *event structure* $(E, \leqslant, \sharp)$ consists of a poset $(E, \leqslant)$ of *events*, the partial order relation expressing *causal dependency*, together with a symmetric irreflexive relation $\sharp$ called *incompatibility* satisfying

- finite causes: for every event e, the set $\{e' \mid e' \leqslant e\}$ is finite,
- hereditary incompatibility: for every events e, e' and e'', $e \sharp e'$ and $e' \leqslant e''$ implies $e \sharp e''$.

EXAMPLES (LABELLED CASE)



$(a \cdot (b + c)$, $a \cdot (b|c)$ and $a \cdot (b \cdot c + c \cdot b)$ resp.)

Relationship Event structures/cubical sets

Configurations of an event structure

- A *configuration* of an event structure $(E, \leqslant, \sharp)$ is a finite downward closed subset of compatible events in E
- An event e is *enabled* at a configuration x if $e \notin x$ and $x \uplus \{e\}$ is a configuration.

"Adjunction" (GETCO 2010)

- 0-cells are the configurations of the event structure with the empty configuration as initial state,
- 1-cells are (x, e), x configuration, e event enabled at x
- 2-cells are the pairs (x, e_1, e_2) where x is a configuration and e_1, e_2 are both enabled at x and such that e_2 is enabled at $x \uplus \{e_1\}$ and e_1 is enabled at $x \uplus \{e_2\}$,

(similar treatment by G. Winskel with asynchronous transition systems)

Configurations of an event structure

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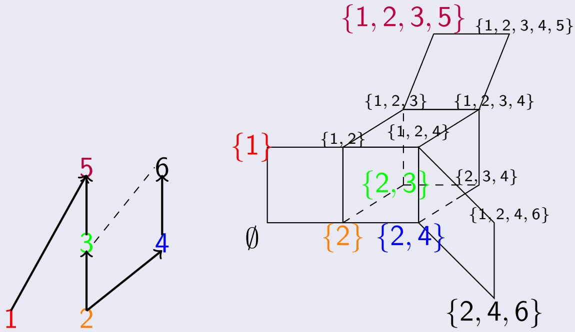
"Adjunction" (GETCO 2010)

- 0-cells are the configurations of the event structure with the empty configuration as initial state,
- 1-cells are (x, e), x configuration, e event enabled at x
- 2-cells:

$$
\begin{tikzcd}
 & x \uplus \{e_1, e_2\} & \\
x \uplus \{e_1\} \arrow[ur, "(x\uplus\{e_1\},e_2)"] & (x, e_1, e_2) & x \uplus \{e_2\} \arrow[ul, "(x\uplus\{e_2\},e_1)"'] \\
 & x \arrow[ul, "(x,e_1)"] \arrow[ur, "(x,e_2)"'] &
\end{tikzcd}
$$

For our (rooted) CAT(0) example



event structure poset of vertices/configurations

(labels are given by the hyperplanes)

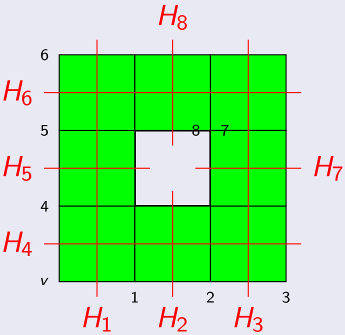
CAT(0) (rooted) cubical complexes and prime event structures

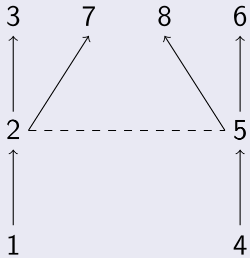
Adjunction (labelled) cubical set/event structure [Goubault/Mimram GETCO 2010]

- We have the adjunction α : PC=(pointed finite precubical sets)$\rightarrow$(event structures)=ES (with γ right adjoint)
- It induces an adjunction α: CC=(rooted finitely generated cubical complexes)$\rightarrow$ ES composing with a geometric realization functor and adjoint
- Result: it restricts to an isomorphism of categories between CAT(0) elements of PC and ES
- Consequence: $\gamma \circ \alpha$ maps rooted (NPC) cubical complexes X to a CAT(0) space $\tilde{X}_v$

$\tilde{X}_v$: AN EXAMPLE

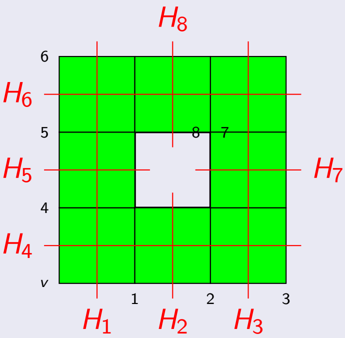
Amounts to counting events with multiplicities

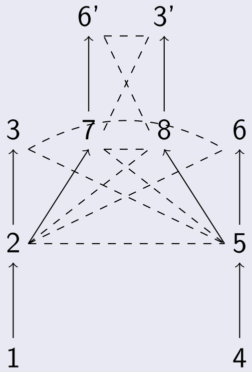




Note: 2 and 5 in conflict corresponds to dead matrix (1 1) in the trace space algorithm!

Amounts to counting events with multiplicities

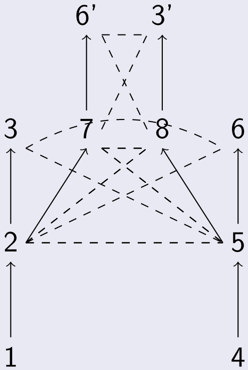


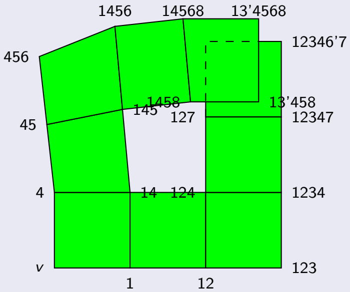


Note: create a copy of event 3 (and 6) as 3 can be reached by 2 or by 8, which are in conflict (is also the unfolding of the corresponding Petri net [current work with Tobias Heindel]!)

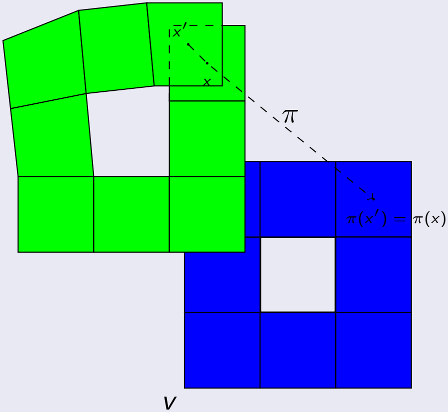
$\tilde{X}_v$: AN EXAMPLE

Amounts to counting events with multiplicities





$\tilde{X}_v$: AN EXAMPLE



THIS IS LISBETH FAJSTRUP'S DICOVERING!

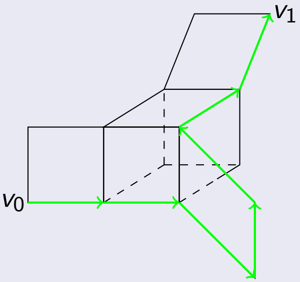
Given a cubical complex X

- A cube path from vertex (i.e. 0-cells) v_0 to v_n in X is a sequence of closed cubes $C = \{C_i\}_{i=1,\dots,n}$ such that,
  - $C_i \cap C_{i+1} = v_i$ a vertex in X
  - C_i is the smallest cube of X containing v_{i-1} and v_i

  The corresponding diagonal path is the piecewise linear path going through $v_0, v_1, \dots, v_n$
- A cube path is normal if $C_{i+1} \cap St(C_i) = v_i$ where $St(C_i)$ is the star of C_i, i.e. all cells, including C_i, which have C_i as a face. The corresponding diagonal path is call a normal diagonal path
- Normal cube paths are discrete counterparts of d_∞-geodesics

Example

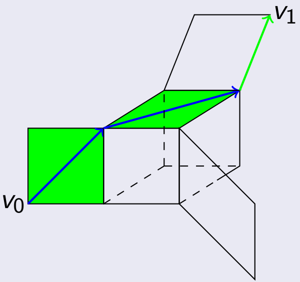
Cube path from v_0 to v_1



In particular, all undirected edge-paths on the 1-skeleton

Example

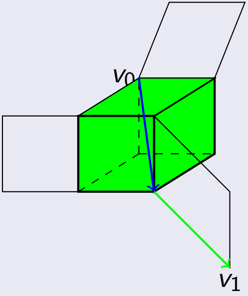
Normal/diagonal cube path from v_0 to v_1



When $v_0 \leqslant v_1$ in $L(X, v)$, normal diagonal cube paths are directed (they can only cross hyperplanes once! corresponds to the intuition a path that goes back in time in one coordinate cannot be a d_∞-geodesic)

Example

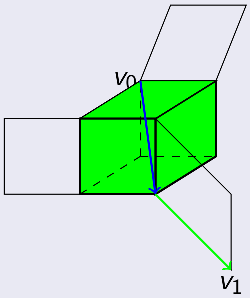
Normal/diagonal cube path from v_0 to v_1



(unlike when $v_0 \not\leq v_1$)

Example

Normal/diagonal cube path from v_0 to v_1



The argument goes through to the NPC case (going to the CAT(0) universal dicovering back and forth)

Pareto optimality (adapted from [Ghrist])

- Given a path $\gamma : [a, b] \to X$, X d_∞-space
- γ is locally Pareto optimal if it is a local d_∞ geodesics
- γ is Pareto optimal if it is a d_∞ geodesics

Pareto equivalence

Two Pareto-optimal paths f and g are Pareto equivalent iff they are homotopic through Pareto-optimal paths which have the same d_∞ arc-length, and the same start and end points (α and β respectively), i.e. iff there exists a continuous function $H : I \times I \to X$ such that:

- $H(0, x) = f(x)$, $H(1, x) = g(x)$
- $H(t, 0) = \alpha$, $H(t, 1) = \beta$
- for all $t \in I$, H_t defined by $H_t(x) = H(t, x)$ is Pareto-optimal

Further results [Ghrist] and consequences

Given an NPC d_∞-space X

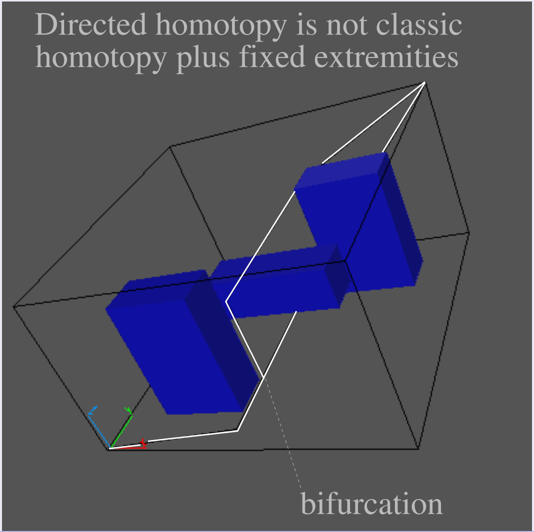
- d_∞-geodesics from v_0 to v_1, $v_0 \leqslant v_1$ in $L(X, v)$, X d_∞-space, are directed
- Hence Pareto-equivalence implies dihomotopy
- All paths are homotopic to a normal cube path [Ghrist 2004] in some subdivision X^N of X
- d_∞-geodesics in d_∞-spaces are Pareto-equivalent to normal cube paths in some subdivision X^N [Ghrist 2004]
- Plus some work to deform all dipaths to normal cube paths (as in Lisbeth's work)

So in fact...

Dipaths modulo homotopy $\equiv$ dipaths modulo dihomotopy in NPC d_∞-spaces

Certainly not true with non-NPC cubical complexes!

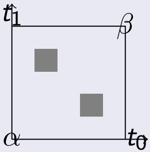
Classical example



- First try (EG, 1992/PEPM 1995):
  - homologies of a certain bicomplex (horizontal=start faces in a chain complex generated by a pre-cubical set, vertical=end faces) define branchings, mergings
  - H_1: total homology relative to endpoints, where we extract only the directed paths, gives π_1 between two endpoints in the plane, for PV models
- Since then many attempts...more or less complicated
- Of course you can consider the homology of the trace space for each pair of endpoints...

NPC cube complexes and directed π_1

Traces in NPC X from α to β



$A = \{\alpha, \beta\}$, $Y = X/A$, supposing there exists a neighborhood U of A such that U has A as deformation retract

Strategy

- Identify $\overrightarrow{\pi}_1(X)(\alpha, \beta)$ as a subset of $\pi_1(Y, \alpha)$, and even of $\pi_1(Y, \alpha)^{ab} \sim H_1(Y) \sim H_1(X) \oplus \mathbb{Z}$ of "minimal length"
- Basically, the Smith Normal form algorithm for calculating $H_1(X)$ has to be completed with Linear Programming

Algorithmically

Generating $\overrightarrow{\pi}_1(X)(\alpha, \beta)$

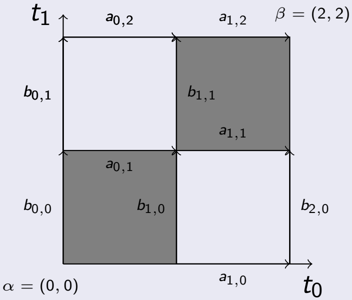
Seen as the subset of dipaths in X generated by generators of $H_1(X)$ and p:

- Let $c_1, \ldots, c_p$ be the generators of $H_1(X)$
- Suppose the edges in $X_1 = \{x_1, \ldots, x_k\}$ are directed, dipaths q are then identified with minimal length elements in $H_1(X) \oplus (p)$ with only positive coefficients in x_i:

$$\begin{array}{rcl} q & = & \sum_{i=1}^{k} \lambda_i x_i \\ q & = & c_0 p + \sum_{i=1}^{l} \mu_j c_j [Im\ \partial_2] \\ \lambda_i & \geqslant & 0 \end{array}$$

Worst-case exponential time (we may have an exponential number of dipaths mod dihomotopy, for a chain of holes)

USUAL EXAMPLE, X IS:



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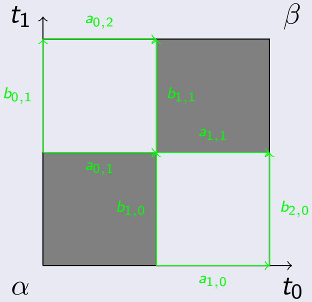
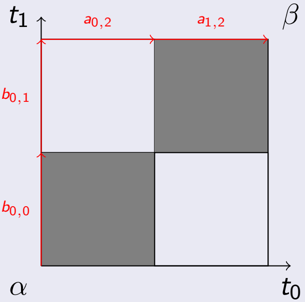


- generators for $H_1(X)$: $t = a_{0,1} + b_{1,1} - a_{0,2} - b_{0,1}$, $z = a_{1,0} + b_{2,0} - a_{1,1} - b_{1,0}$

USUAL EXAMPLE, X IS:



- generators for $H_1(X)$: $t = a_{0,1} + b_{1,1} - a_{0,2} - b_{0,1}$, $z = a_{1,0} + b_{2,0} - a_{1,1} - b_{1,0}$
- extra-generator for $H_1(X/\{\alpha, \beta\})$:

$$x = b_{0,0} + b_{0,1} + a_{0,2} + a_{1,2}$$

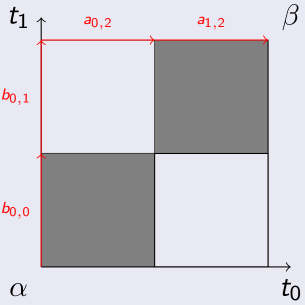
FINALLY

- Looking at the extremal points/rays of the polyhedron given by constraints $\partial A_{00} = -a_{0,0} + a_{0,1} + b_{00} - b_{10} = 0$, $\partial A_{11} = -a_{11} + a_{12} + b_{11} - b_{21} = 0$ and $\mu_0 x + \mu_1 z + \mu_2 t$ has positive coefficients λ_i in the (directed) edges, with minimal length $\sum_i \lambda_i$ (=4 here!): i.e. such that:

$$\begin{array}{rcl} \mu_1 & \leqslant & 0 \\ \mu_2 & \leqslant & 0 \\ \ldots & & \end{array}$$

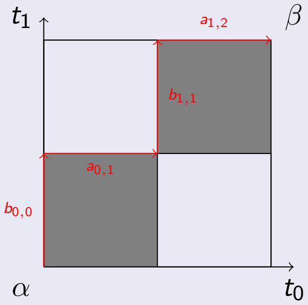
  I.e. for all extremal points of the above polyhedron
- We find x, $x - t$, and $x - t - z - A_{11} - A_{00}$

Finally

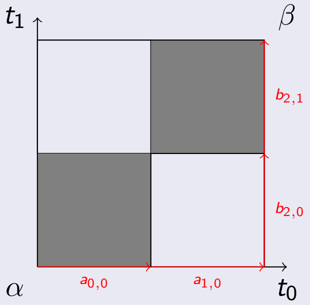


Dihomotopy classes from α to β

Finally



FINALLY



Conclusion and future work

Homological calculations

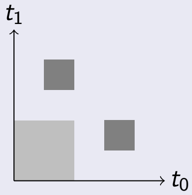
- Can that be competitive with trace space calculations in the NPC case? Using RedHom techniques?
- Higher dimensional homology and "$\overrightarrow{\pi}_n$" in the NPC case?
- Persistent homology for varying endpoints?

Generalizations to this approach

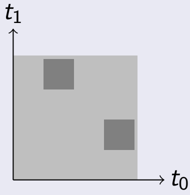
- For more general cubical complexes? (non NPC)
- Some directions: unfoldings of (unsafe) Petri nets (with Tobias Heindel), combinatorics of more general hyperplane arrangements in the dicoverings in the non NPC case (and links with the trace space algorithm)

Idea:

- Persistent homology, i.e. with just a filtration:
  - Use of "global time" evolving on the diagonal for instance



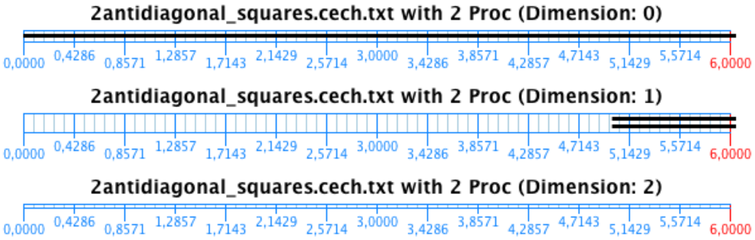
Trivial homology



Suddenly, H_1 has two generators

  - Sub-level sets as filtration
- Multidimensional persistent homology:
  - For PV programs, natural choice is the tuple of local times on each process! Exact for NPC spaces!

Some hints on the persistent homology approach



(some other experiments using RedHom or Javaplex to be reported soon)

Thanks for your attention!

http://acat.lix.polytechnique.fr