Finite Frame Varieties: A Tutorial

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Quick Outline

1. Definitions and Notation
2. Nonsingular points of FUNTF varieties
3. Parameterizations of FUNTF varieties
4. Connectivity and irreducibility of FUNTF Varieties

Highly Redundant Definitions

A collection of vectors $\{f_i\}_{i=1}^N \subset \mathbb{R}^d$ is a **frame** if there are constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sum_{i=1}^N |\langle x, f_i\rangle|^2 \leq B\|x\|^2 \text{ for all } x \in \mathbb{R}^d.$$

Highly Redundant Definitions

If we can take $A = B$, then we say that the frame is **tight**.

Highly Redundant Definitions

If we have that $\|f_i\| = 1$ for all i in the N-set $[N]$, then we say that the frame is **unit-norm**.

Highly Redundant Definitions

If a finite frame is both unit-norm and tight, we say that it is a **FUNTF**.

Identification with Matrices

In this talk, we identify an indexed finite frame with the matrix

$$F = \begin{pmatrix} \langle f_1, e_1\rangle & \langle f_2, e_1\rangle & \cdots & \langle f_N, e_1\rangle \\ \langle f_1, e_2\rangle & \langle f_2, e_2\rangle & \cdots & \langle f_N, e_2\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_1, e_d\rangle, & \langle f_2, e_d\rangle & \cdots & \langle f_N, e_d\rangle \end{pmatrix} \in M_{d,N}$$

where $\{e_i\}_{i\in[d]}$ is the standard orthonormal basis of $\mathbb{R}^d$.

Identification with Matrices

We also write $F = [f_1\ f_2 \cdots f_N]$. Letting F^* denote the transpose of this matrix, we have the **frame operator** of F

$$FF^* = \sum_{i=1}^{N} f_i f_i^* = \sum_{i=1}^{N} \begin{pmatrix} \langle f_i, e_1\rangle\langle f_i, e_1\rangle & \langle f_i, e_1\rangle\langle f_i, e_2\rangle & \cdots & \langle f_i, e_1\rangle\langle f_i, e_d\rangle \\ \langle f_i, e_2\rangle\langle f_i, e_1\rangle & \langle f_i, e_2\rangle\langle f_i, e_2\rangle & \cdots & \langle f_i, e_2\rangle\langle f_i, e_d\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_i, e_d\rangle\langle f_i, e_1\rangle & \langle f_i, e_d\rangle\langle f_i, e_2\rangle & \cdots & \langle f_i, e_d\rangle\langle f_i, e_d\rangle \end{pmatrix}$$

and the **Grammian** of F

$$F^*F = \begin{pmatrix} \langle f_1, f_1\rangle & \langle f_1, f_2\rangle & \cdots & \langle f_1, f_N\rangle \\ \langle f_2, f_1\rangle & \langle f_2, f_2\rangle & \cdots & \langle f_2, f_N\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle f_N, f_1\rangle & \langle f_N, f_2\rangle & \cdots & \langle f_N, f_N\rangle \end{pmatrix}.$$

Identification with Matrices

Lemma. *Let* $F \in M_{N \times d}$.

- F *is a tight frame with frame bounds* $A = B = \frac{N}{d}$ *if and only if*

$$F \in St_{d,N} = \{X \in M_{d,N} : XX^* = \frac{N}{d} I_d\}.$$

- F *is unit-norm if and only if*

$$F \in \mathbb{T}_{d,N} = \{X = [x_1 \cdots x_N] \in M_{d,N} : \|x_i\| = 1 \textit{ for all } i \in [N]\}$$

- F *is a FUNTF if and only if*

$$F \in St_{d,N} \cap \mathbb{T}_{d,N} = \mathcal{F}_{d,N}.$$

Our Standard Examples

$$
\begin{aligned}
\Phi &= \left[\phi_1\ \phi_2\ \phi_3\ \phi_4\ \phi_5\ \phi_6\right] \\
&= \begin{pmatrix} 1 & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{6}}{6} \\ 0 & \frac{\sqrt{3}}{3} & 1 & 0 & 0 & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{2}}{2} & 1 & \frac{\sqrt{6}}{6} \end{pmatrix}
\end{aligned}
$$

Our Standard Examples

$$\Xi = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Spaces of FUNTFs as Algebraic Varieties

An **algebraic variety** is the zero set of a system of polynomials.

Spaces of FUNTFs as Algebraic Varieties

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix} \text{ is in } St_{3,6} \text{ if and only if}$$

$$\begin{aligned}
x_{11}^2 + x_{12}^2 + x_{13}^2 + x_{14}^2 + x_{15}^2 + x_{16}^2 &= 2 \\
x_{21}^2 + x_{22}^2 + x_{23}^2 + x_{24}^2 + x_{25}^2 + x_{26}^2 &= 2 \\
x_{31}^2 + x_{32}^2 + x_{33}^2 + x_{34}^2 + x_{35}^2 + x_{36}^2 &= 2 \\
x_{11}x_{21} + x_{12}x_{22} + x_{13}x_{23} + x_{14}x_{24} + x_{15}x_{25} + x_{16}x_{26} &= 0 \\
x_{11}x_{31} + x_{12}x_{32} + x_{13}x_{33} + x_{14}x_{34} + x_{15}x_{35} + x_{16}x_{36} &= 0 \\
x_{21}x_{31} + x_{22}x_{32} + x_{23}x_{33} + x_{24}x_{34} + x_{25}x_{35} + x_{26}x_{36} &= 0
\end{aligned}$$

Spaces of FUNTFs as Algebraic Varieties

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{14} & x_{15} & x_{16} \\ x_{21} & x_{22} & x_{23} & x_{24} & x_{25} & x_{26} \\ x_{31} & x_{32} & x_{33} & x_{34} & x_{35} & x_{36} \end{pmatrix} \text{ is in } \mathbb{T}_{3,6} \text{ if and only if}$$

$$\begin{aligned}
x_{11}^2 + x_{21}^2 + x_{31}^2 &= 1 \\
x_{12}^2 + x_{22}^2 + x_{32}^2 &= 1 \\
x_{13}^2 + x_{23}^2 + x_{33}^2 &= 1 \\
x_{14}^2 + x_{24}^2 + x_{34}^2 &= 1 \\
x_{15}^2 + x_{25}^2 + x_{35}^2 &= 1 \\
x_{16}^2 + x_{26}^2 + x_{36}^2 &= 1
\end{aligned}$$

Basic Question: Local Structure

What are the non-singular points of $\mathcal{F}_{d,N}$?

Basic Question: Local Structure

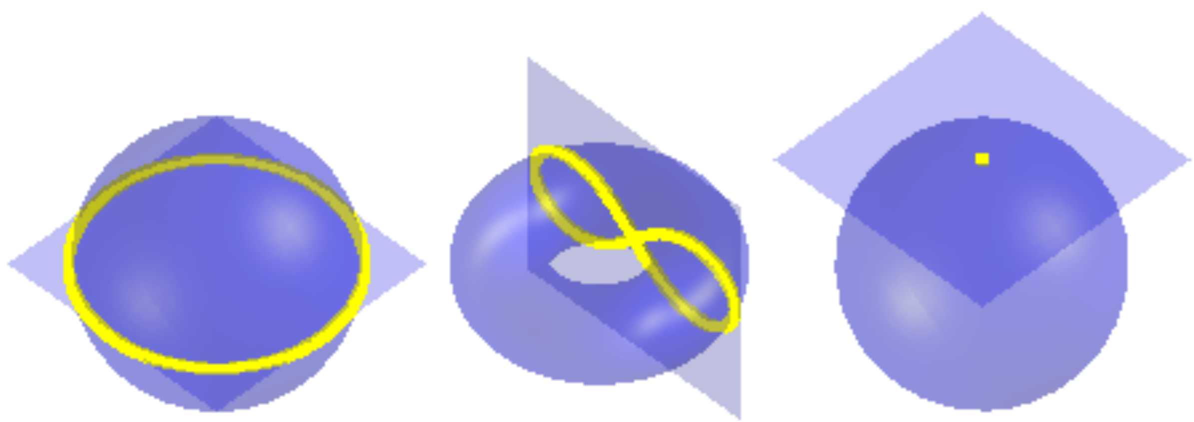
If $F \in \mathcal{F}_{d,N}$ is non-singular, then what is $T_F \mathcal{F}_{d,N}$?

From Global to Local

$$\mathcal{F}_{d,N} = \mathrm{St}_{d,N} \cap \mathbb{T}_{d,N}, \text{ so } T_F\mathcal{F}_{d,N} = T_F\mathrm{St}_{d,N} \cap T_F\mathbb{T}_{d,N}?$$

Transversal Intersections

Definition. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are two smooth manifolds embedded in the same smooth manifold $\mathcal{K}$, and suppose $X \in \mathcal{M} \cap \mathcal{N}$. Then we say $\mathcal{M}$ and $\mathcal{N}$* **intersect transversally** *in $\mathcal{K}$ at X if $T_X\mathcal{K} = T_X\mathcal{M} + T_X\mathcal{N}$ (where $+$ is the Minkowski sum).*



Transversal Intersections

Proposition. *Suppose $\mathcal{M}$ and $\mathcal{N}$ intersect transversally in $\mathcal{K}$ at X. Then X is a nonsingular point of $\mathcal{M} \cap \mathcal{N}$ and*

$$T_X(\mathcal{M} \cap \mathcal{N}) = T_X\mathcal{M} \cap T_X\mathcal{N}.$$

Characterizations of Singular Points

We may therefore characterize the singular points of $\mathcal{F}_{d,N}$ as the points where the intersection of $\mathrm{St}_{d,N}$ and $\mathbb{T}_{N,d}$ fails to be transversal.

One important point is that $\mathrm{St}_{d,N}$ and $\mathbb{T}_{d,N}$ are both contained in the Hilbert-Schmidt sphere

$$\mathcal{S}_{d,N} = \{X \in M_{d,N} : \mathrm{trace}(XX^*) = N\}.$$

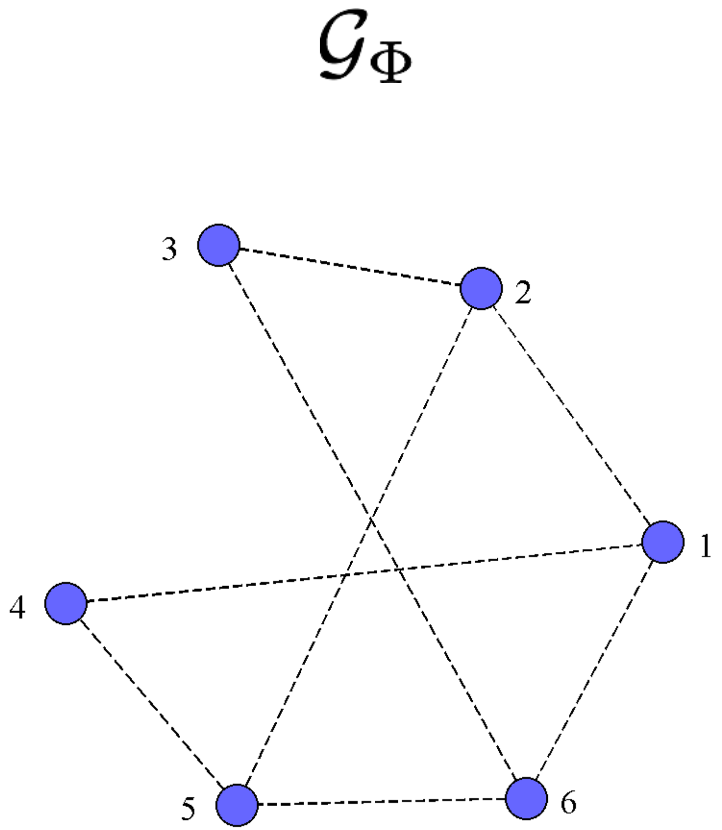
Transversality of the intersection is therefore relative to $\mathcal{S}_{d,N}$ instead of $M_{d,N}$.

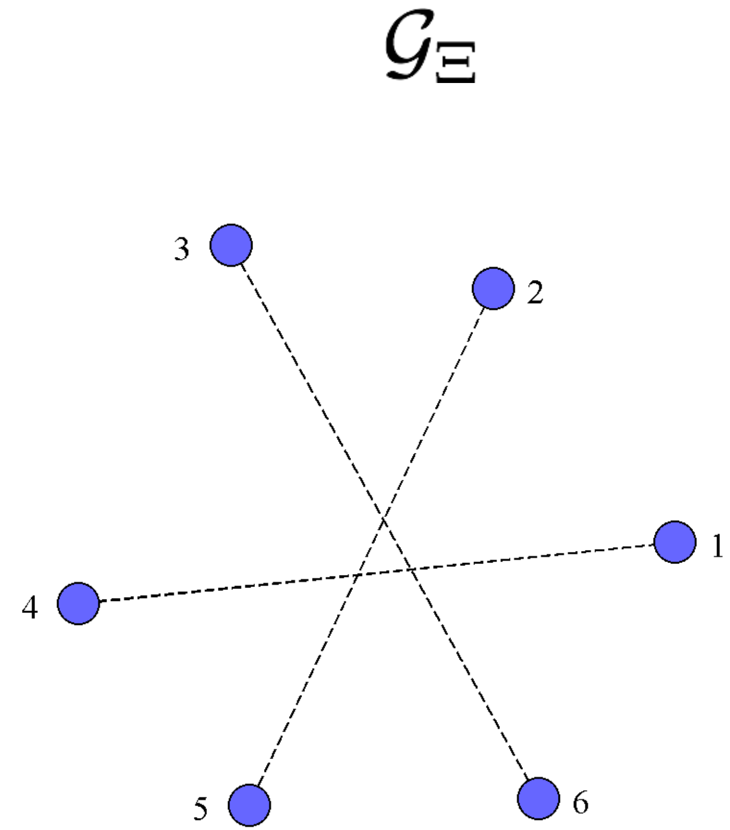
Correlation Networks

Definition. *For a frame* $F = [f_1\ f_2\ \cdots\ f_N] \in M_{d,N}$, *the* **correlation network** *is the symmetric graph* $\mathcal{G}_F = (V, E)$ *with vertices* $V = \{1, 2, \ldots, N\}$ *and edge set*

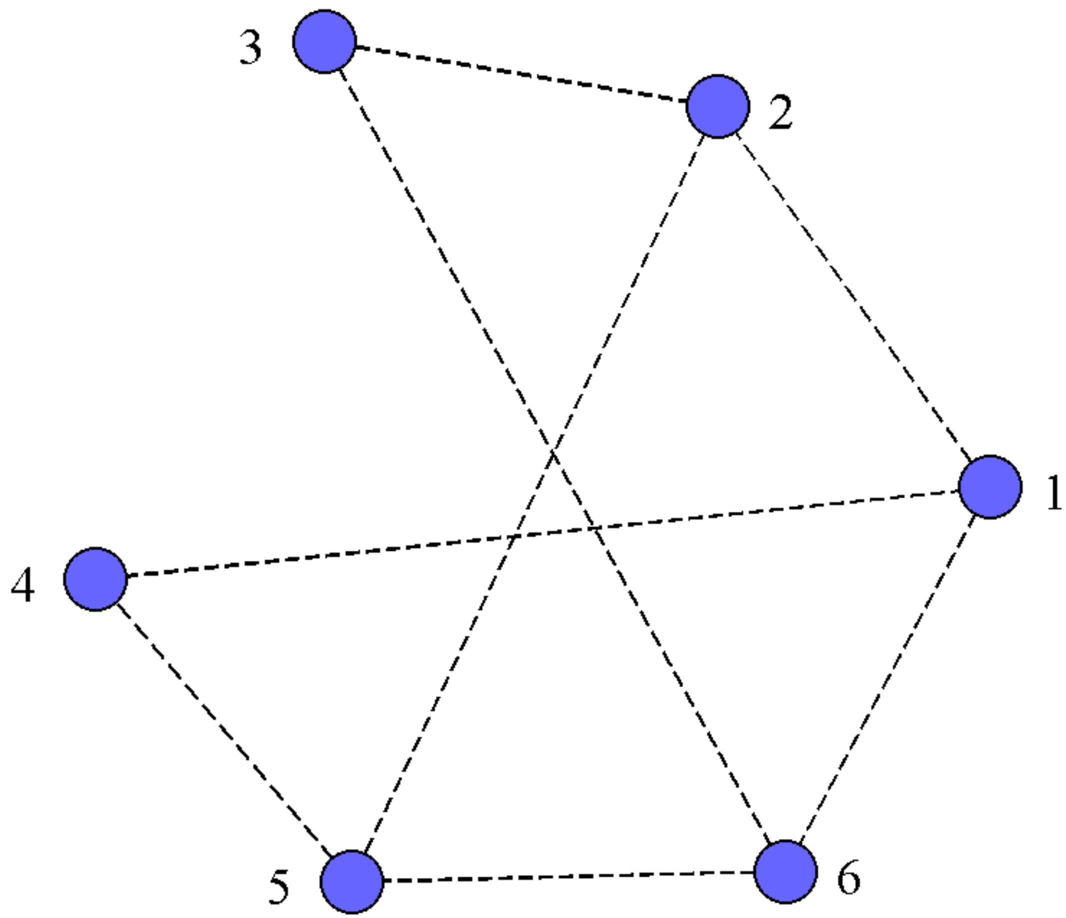
$$E = \{(i, j) : \langle f_i, f_j \rangle \neq 0\}.$$

Example

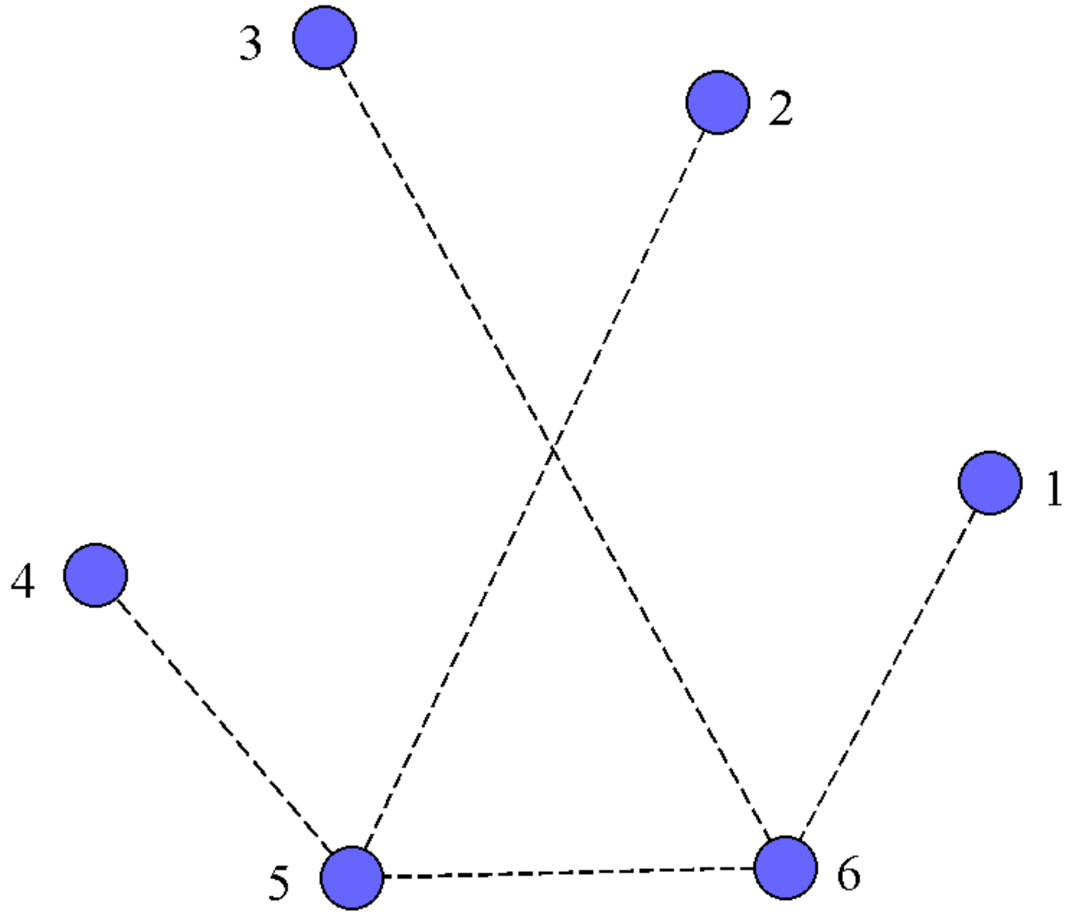




Example



Example



Orthodecomposable Frames

Proposition. *The correlation network* $\mathcal{G}_F$ *is connected if and only if* F *cannot be partitioned into two non-trivial subsets of matrices with orthogonal column spaces. If* F *does admit such a partition, we say that* F *is* **orthodecomposable**.

Locally Transversal Intersections of Tori and Stiefel Manifolds

Theorem. *Suppose $N \geq d \geq 2$. The manifolds $\mathbb{T}_{d,N}$ and $St_{d,N}$ intersect transversally in $\mathcal{S}_{d,N}$ at $F \in \mathcal{F}_{d,N}$ if and only if F is not orthodecomposable. Moreover, the local dimension of $\mathcal{F}_{d,N}$ around such an F is given by*

$$(d-1)N + \left(dN - \binom{d+1}{2}\right) - (dN-1) = (d-1)N - \binom{d+1}{2} + 1.$$

When FUNTF Spaces are Manifolds

Proposition. *The variety $\mathcal{F}_{d,N}$ is a manifold if and only if N and d are relatively prime.*

Basic Question: Local Coordinates

Can we write down local parameterizations of $\mathcal{F}_{d,N}$?

FUNTF Equations are Locally Solvable

Idea: Freely articulate $N-d$ vectors in the frame and then the remaining d vectors react to preserve tightness.

FUNTF Equations are Locally Solvable

$$\widetilde{F} = [\widetilde{f_1}\ \widetilde{f_2}\ \widetilde{f_3}\ \widetilde{f_4}\ \widetilde{f_5}\ \widetilde{f_6}] = [\widetilde{F_1}\widetilde{F_2}]$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}\widetilde{F}^* = \widetilde{F}_1\widetilde{F}_1^* + \widetilde{F}_2\widetilde{F}_2^* = 2I_3$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2\widetilde{F}_2^* = \overbrace{2I_3 - \widetilde{F}_1\widetilde{F}_1^*}^{\widetilde{H}}$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2 \widetilde{F}_2^* \widetilde{H}^{-1} = I_3$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2^* \widetilde{H}^{-1} = \widetilde{F}_2^{-1}$$

FUNTF Equations are Locally Solvable

$$\widetilde{F}_2^* \widetilde{H}^{-1} \widetilde{F}_2 = I_3$$

FUNTF Equations are Locally Solvable

$$
\begin{aligned}
\widetilde{f}_4^* H^{-1} \widetilde{f}_4 &= 1 \\
\widetilde{f}_4^* \widetilde{f}_4 &= 1
\end{aligned}
$$

FUNTF Equations are Locally Solvable

$$\begin{aligned} f_4^*(2I_4 - F_1F_1^*)^{-1}g &= 0 \\ f_4^*g &= 0 \\ \Rightarrow \widetilde{f}_4^*g &= \theta \end{aligned}$$

FUNTF Equations are Locally Solvable

$$
\begin{aligned}
\widetilde{f}_4^* \widetilde{H}^{-1} \widetilde{f}_4 &= 1 \\
\widetilde{f}_4^* \widetilde{f}_4 &= 1 \\
\widetilde{f}_4^* g &= \theta \\
\widetilde{f}_4 &= x H^{-1} f_4 + y f_4 + \theta g
\end{aligned}
$$

Solving a System of Two Quadratics and a Linear Equation

$$\begin{aligned}(f_4^* H^{-1}\widetilde{H}^{-1}H^{-1}f_4)x^2 + 2f_4^* H^{-1}\widetilde{H}^{-1}(yf_4+\theta g)x + (yf_4+\theta g)^*\widetilde{H}^{-1}(yf_4+\theta g) - 1 &= 0\\ (f_4^* H^{-2}f_4)x^2 + 2yf_4^* H^{-1}f_4 x + y^2 + \theta^2 - 1 &= 0\end{aligned}$$

Solving a System of Two Quadratics and a Linear Equation

Proposition. *The system*

$$\begin{aligned} \alpha_2 x^2 + \alpha_1 x + \alpha_0 &= 0 \\ \beta_2 x^2 + \beta_1 x + \beta_0 &= 0 \end{aligned}$$

where α_2 and β_2 are non-zero admits a solution if and only if the Bézout determinant vanishes:

$$(\alpha_2\beta_1 - \alpha_1\beta_2)(\alpha_1\beta_0 - \alpha_0\beta_1) - (\alpha_2\beta_0 - \alpha_0\beta_2)^2 = 0.$$

Solving a System of Two Quadratics and a Linear Equation

This yields a quartic in the variable y, which has an explicit, but complicated solution.

Eigensteps: Cleaner Local Coordinates

Let $\widetilde{\lambda} : M_{d,d} \to \mathbb{R}^d$ be such that

$$\widetilde{\lambda}(X) = \begin{pmatrix} \lambda_1(X) \\ \lambda_2(X) \\ \vdots \\ \lambda_d(X) \end{pmatrix}$$

where $\{\lambda_i(X)\}_{i=1}^d$ are the eigenvalues of X counting multiplicity, and put in non-increasing order.

Eigensteps: Cleaner Local Coordinates

Define the **eigensteps map**, $\lambda : M_{d,N} \to M_{d,N}$, by

$$\lambda(X) = [\widetilde{\lambda}(X_1 X_1^*)\ \widetilde{\lambda}(X_2 X_2^*) \cdots \widetilde{\lambda}(XX^*)]$$

where $X_n = [x_1\ x_2 \cdots x_n]$ for all n and $X = X_N$

Eigensteps: Necessary Conditions

- Trace condition:

$$\sum_{i=1}^{d} \lambda_i^{(n+1)} = \|x_{n+1}\|^2 + \sum_{i=1}^{d} \lambda_i^{(n+1)}$$

- Interlacing inequalties:

$$\lambda_1^{(n+1)} \geq \lambda_1^{(n)} \geq \lambda_2^{(n+1)} \geq \cdots \geq \lambda_d^{(n+1)} \geq \lambda_d^{(n)}$$

Eigensteps: Also Sufficient

Lemma. $\Delta_{d,N} = \lambda(\mathcal{F}_{d,N})$ *is the polytope determined by the interlacing inequalities and trace conditions.*

Example

$$\lambda(\Phi) = \begin{pmatrix} 1 & (3+\sqrt{3})/3 & (3+\sqrt{6})/3 & 2 & 2 & 2 \\ 0 & (3-\sqrt{3})/3 & 1 & (6+\sqrt{6})/6 & 2 & 2 \\ 0 & 0 & (3-\sqrt{6})/3 & (6-\sqrt{6})/6 & 1 & 2 \end{pmatrix}$$

Lifting Eigensteps

Proposition. *Let* $\lambda(F) \in \Delta^\circ_{3,6}$*, where* $F \in \mathcal{F}_{3,6}$ *and* $\Delta^\circ_{3,6}$ *denotes the interior of* $\Delta_{3,6}$*. Then there are sequences of vector-valued functions* $v_k : \Delta^\circ_{3,6} \to \mathbb{R}^3$ *and* $w_k : \Delta^\circ_{3,6} \to \mathbb{R}^3$*, a sequence of matrix-valued functions* $W_k : \Delta^\circ_{3,6} \to M_{3,3}$*, and sequences of orthogonal matrices* V_k*,* P_k*, and* Q_k *so that when we define the sequences*

$$\begin{aligned}
U_1(U,\mu) &= U \\
\phi_1(U,\mu) &= U_1(U,\mu)e_1 \\
\phi_{k+1}(U,\mu) &= U_k(U,\mu)V_kP_k^*v_k(\mu) \\
U_{k+1}(U,\mu) &= U_k(U,\mu)V_kP_k^*W_k(\mu)Q_k
\end{aligned}$$

for $k = 1, 2, 3, 4, 5$ *and all* $(U,\mu) \in \mathcal{O}(3) \times \Delta^\circ_{3,6}$*, then*

$$\Phi(U,\mu) = [\phi_1(U,\mu)\ \phi_2(U,\mu)\ \phi_3(U,\mu)\ \phi_4(U,\mu)\ \phi_5(U,\mu)\ \phi_6(U,\mu)] \in \mathcal{F}_{d,N}$$

satisfies $\lambda(\Phi(U,\mu)) = \mu$ *and* $\Phi(\mathcal{Q}(F), \lambda(F)) = F$ *where* $\mathcal{Q}(F)$ *is* Q *from the* QR *decomposition of the first* 3 *columns of* F*.*

Example: Coordinates from Eigensteps

k	$v_k(\lambda)$	$w_k(\lambda)$	$W_k(\lambda)$
1	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{11}-\lambda_{12})(\lambda_{11}-\lambda_{22})(\lambda_{11}-\lambda_{32})}{(\lambda_{11}-\lambda_{21})(\lambda_{11}-\lambda_{31})}}\\ \sqrt{-\frac{(\lambda_{21}-\lambda_{12})(\lambda_{21}-\lambda_{22})}{(\lambda_{21}-\lambda_{11})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{12}-\lambda_{11})(\lambda_{12}-\lambda_{21})(\lambda_{12}-\lambda_{31})}{(\lambda_{12}-\lambda_{22})(\lambda_{12}-\lambda_{32})}}\\ \sqrt{\frac{(\lambda_{22}-\lambda_{11})(\lambda_{22}-\lambda_{21})}{(\lambda_{21}-\lambda_{11})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\frac{v_{11}(\lambda)w_{11}(\lambda)}{\lambda_{12}-\lambda_{11}} & \frac{v_{11}(\lambda)w_{21}(\lambda)}{\lambda_{22}-\lambda_{11}} & 0\\ \frac{v_{21}(\lambda)w_{11}(\lambda)}{\lambda_{12}-\lambda_{21}} & \frac{v_{21}(\lambda)w_{21}(\lambda)}{\lambda_{22}-\lambda_{21}} & 0\\ 0 & 0 & 1\end{pmatrix}$
2	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{12}-\lambda_{13})(\lambda_{12}-\lambda_{23})(\lambda_{12}-\lambda_{33})}{(\lambda_{12}-\lambda_{22})(\lambda_{12}-\lambda_{32})}}\\ \sqrt{-\frac{(\lambda_{22}-\lambda_{13})(\lambda_{22}-\lambda_{23})(\lambda_{22}-\lambda_{33})}{(\lambda_{22}-\lambda_{12})(\lambda_{22}-\lambda_{32})}}\\ \sqrt{-\frac{(\lambda_{32}-\lambda_{13})(\lambda_{32}-\lambda_{23})(\lambda_{32}-\lambda_{33})}{(\lambda_{32}-\lambda_{12})(\lambda_{32}-\lambda_{22})}}\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{13}-\lambda_{12})(\lambda_{13}-\lambda_{22})(\lambda_{13}-\lambda_{32})}{(\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{33})}}\\ \sqrt{\frac{(\lambda_{23}-\lambda_{12})(\lambda_{23}-\lambda_{22})(\lambda_{23}-\lambda_{32})}{(\lambda_{23}-\lambda_{13})(\lambda_{23}-\lambda_{33})}}\\ \sqrt{\frac{(\lambda_{33}-\lambda_{12})(\lambda_{33}-\lambda_{22})(\lambda_{33}-\lambda_{32})}{(\lambda_{33}-\lambda_{13})(\lambda_{33}-\lambda_{23})}}\end{pmatrix}$	$\begin{pmatrix}\frac{v_{12}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{12}} & \frac{v_{12}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{12}} & \frac{v_{12}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{12}}\\ \frac{v_{22}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{22}} & \frac{v_{22}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{22}} & \frac{v_{22}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{22}}\\ \frac{v_{32}(\lambda)w_{12}(\lambda)}{\lambda_{13}-\lambda_{32}} & \frac{v_{32}(\lambda)w_{22}(\lambda)}{\lambda_{23}-\lambda_{32}} & \frac{v_{32}(\lambda)w_{32}(\lambda)}{\lambda_{33}-\lambda_{32}}\end{pmatrix}$
3	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{13}-\lambda_{14})(\lambda_{13}-\lambda_{24})(\lambda_{13}-\lambda_{34})}{(\lambda_{13}-\lambda_{23})(\lambda_{13}-\lambda_{33})}}\\ \sqrt{-\frac{(\lambda_{23}-\lambda_{14})(\lambda_{23}-\lambda_{24})(\lambda_{23}-\lambda_{34})}{(\lambda_{23}-\lambda_{13})(\lambda_{23}-\lambda_{33})}}\\ \sqrt{-\frac{(\lambda_{33}-\lambda_{14})(\lambda_{33}-\lambda_{24})(\lambda_{33}-\lambda_{34})}{(\lambda_{33}-\lambda_{13})(\lambda_{33}-\lambda_{23})}}\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{14}-\lambda_{13})(\lambda_{14}-\lambda_{23})(\lambda_{14}-\lambda_{33})}{(\lambda_{14}-\lambda_{24})(\lambda_{14}-\lambda_{34})}}\\ \sqrt{\frac{(\lambda_{24}-\lambda_{13})(\lambda_{24}-\lambda_{23})(\lambda_{24}-\lambda_{33})}{(\lambda_{24}-\lambda_{14})(\lambda_{24}-\lambda_{34})}}\\ \sqrt{\frac{(\lambda_{34}-\lambda_{13})(\lambda_{34}-\lambda_{23})(\lambda_{34}-\lambda_{33})}{(\lambda_{34}-\lambda_{14})(\lambda_{34}-\lambda_{24})}}\end{pmatrix}$	$\begin{pmatrix}\frac{v_{13}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{13}} & \frac{v_{13}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{13}} & \frac{v_{13}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{13}}\\ \frac{v_{23}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{23}} & \frac{v_{23}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{23}} & \frac{v_{23}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{23}}\\ \frac{v_{33}(\lambda)w_{13}(\lambda)}{\lambda_{14}-\lambda_{33}} & \frac{v_{33}(\lambda)w_{23}(\lambda)}{\lambda_{24}-\lambda_{33}} & \frac{v_{33}(\lambda)w_{33}(\lambda)}{\lambda_{34}-\lambda_{33}}\end{pmatrix}$
4	$\begin{pmatrix}\sqrt{-\frac{(\lambda_{24}-\lambda_{15})(\lambda_{24}-\lambda_{25})(\lambda_{24}-\lambda_{35})}{(\lambda_{24}-\lambda_{14})(\lambda_{24}-\lambda_{34})}}\\ \sqrt{-\frac{(\lambda_{34}-\lambda_{15})(\lambda_{34}-\lambda_{25})(\lambda_{34}-\lambda_{35})}{(\lambda_{34}-\lambda_{14})(\lambda_{34}-\lambda_{24})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\sqrt{\frac{(\lambda_{25}-\lambda_{24})(\lambda_{25}-\lambda_{34})}{(\lambda_{25}-\lambda_{35})}}\\ \sqrt{\frac{(\lambda_{35}-\lambda_{14})(\lambda_{35}-\lambda_{24})(\lambda_{35}-\lambda_{34})}{(\lambda_{35}-\lambda_{15})(\lambda_{35}-\lambda_{25})}}\\ 0\end{pmatrix}$	$\begin{pmatrix}\frac{v_{14}(\lambda)w_{14}(\lambda)}{\lambda_{25}-\lambda_{24}} & \frac{v_{14}(\lambda)w_{24}(\lambda)}{\lambda_{35}-\lambda_{24}} & 0\\ \frac{v_{24}(\lambda)w_{14}(\lambda)}{\lambda_{25}-\lambda_{34}} & \frac{v_{24}(\lambda)w_{24}(\lambda)}{\lambda_{35}-\lambda_{34}} & 0\\ 0 & 0 & 1\end{pmatrix}$
5	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

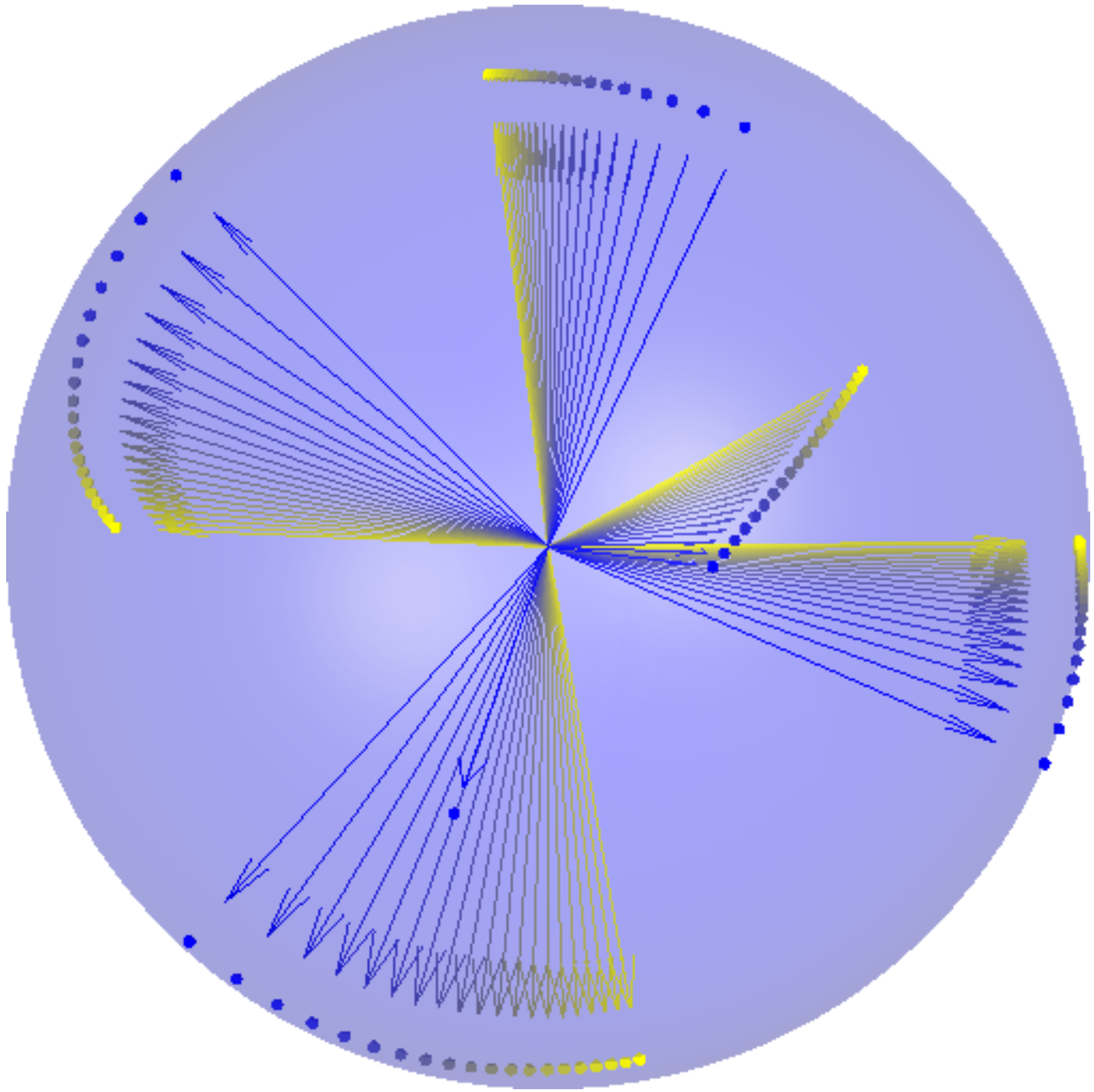
Example: Coordinates from Eigensteps

$$\mathcal{Q}(\Phi) \quad = \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{2}/2 & \sqrt{2}/2 \\ 0 & \sqrt{2}/2 & -\sqrt{2}/2 \end{pmatrix},$$

Example: Coordinates from Eigensteps

k	V_k	P_k	Q_k
1	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
2	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
3	$\begin{pmatrix}1&0&0\\0&-1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$
4	$-\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}$
5	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$	$\begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$	$\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$

Eigenstep Trajectories



Frame Homotopy Problem

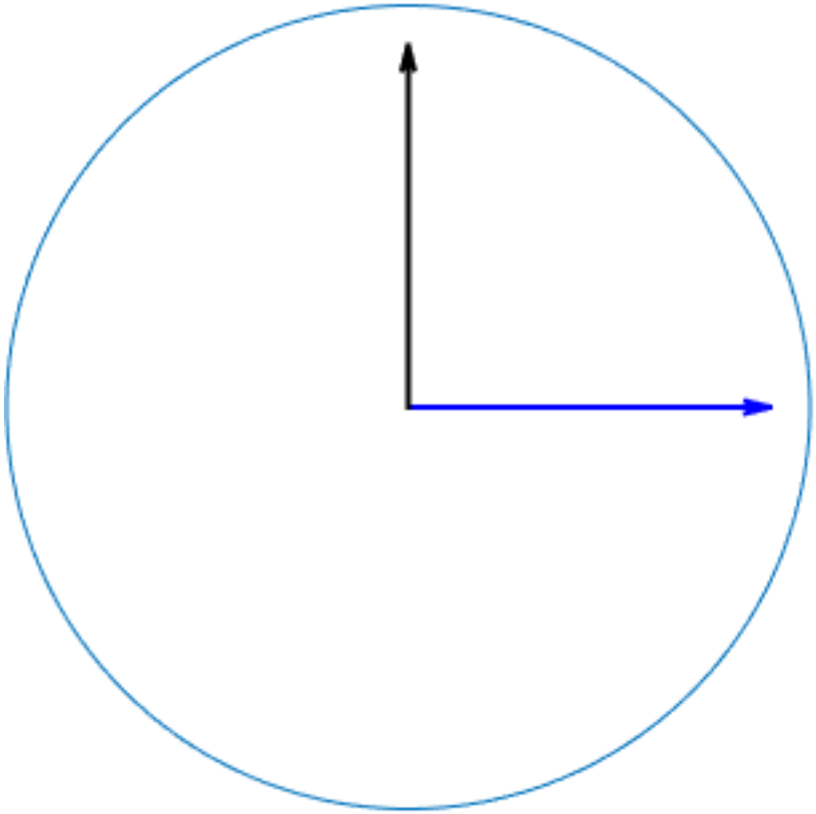
Find all pairs (N, d) such that $\mathcal{F}_{d,N}$ path-connected.

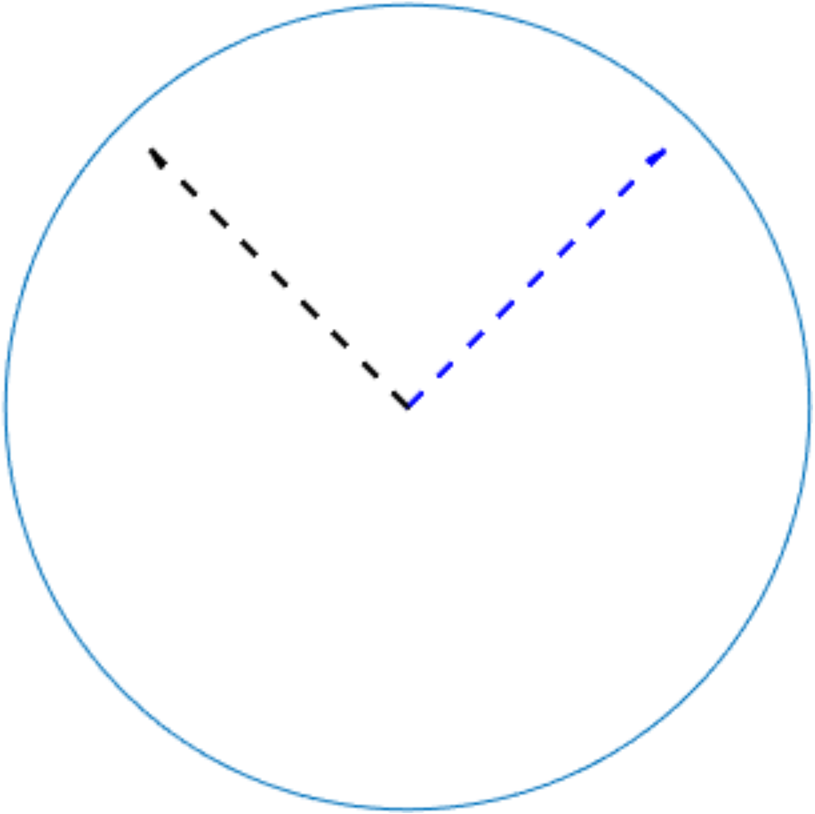
Idea: Identify "Hubs" by Eigensteps and Connect

The "hub" for $\mathcal{F}_{3,6}$ is the union of two orthonormal bases indicated by the eigensteps

$$\begin{pmatrix} 1 & 1 & 1 & 2 & 2 & 2 \\ 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 1 & 2 \end{pmatrix}$$

Swapping Vectors between ONBs





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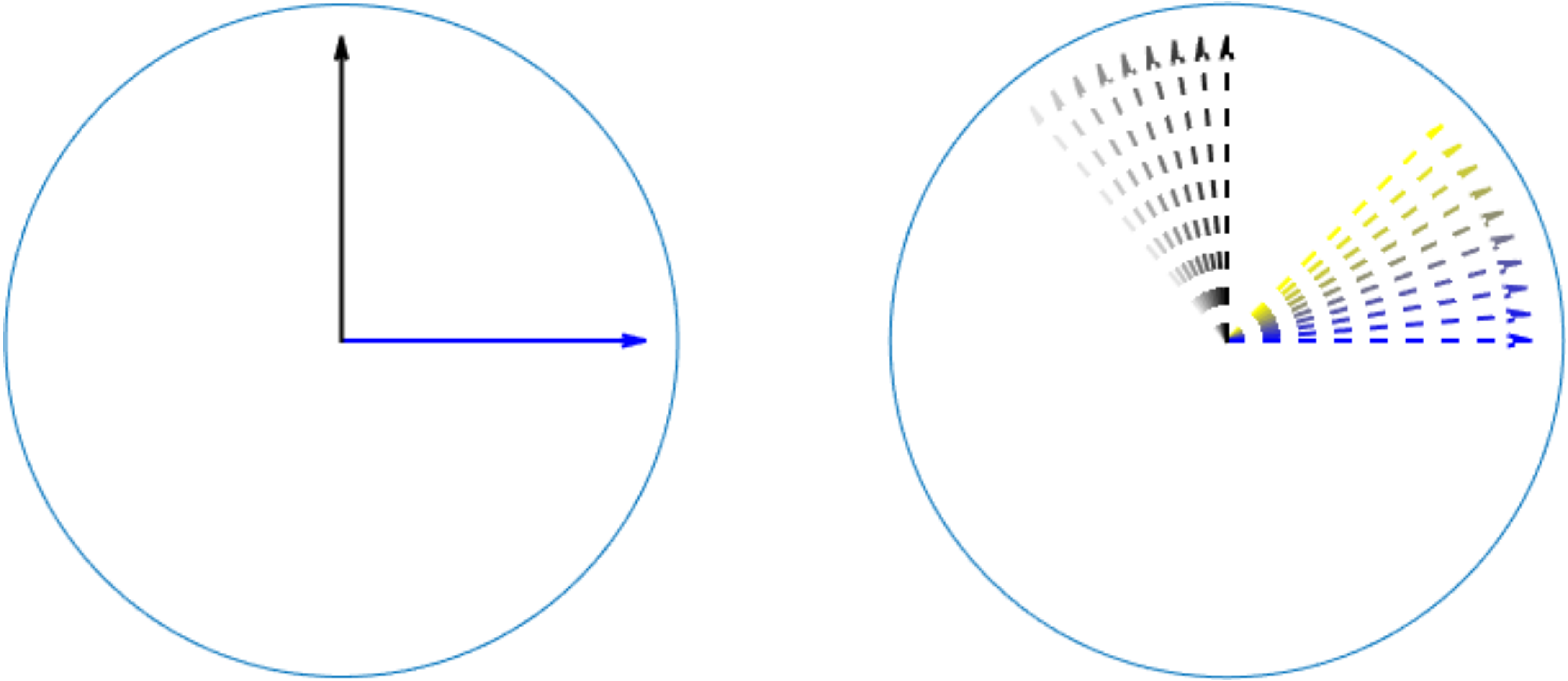

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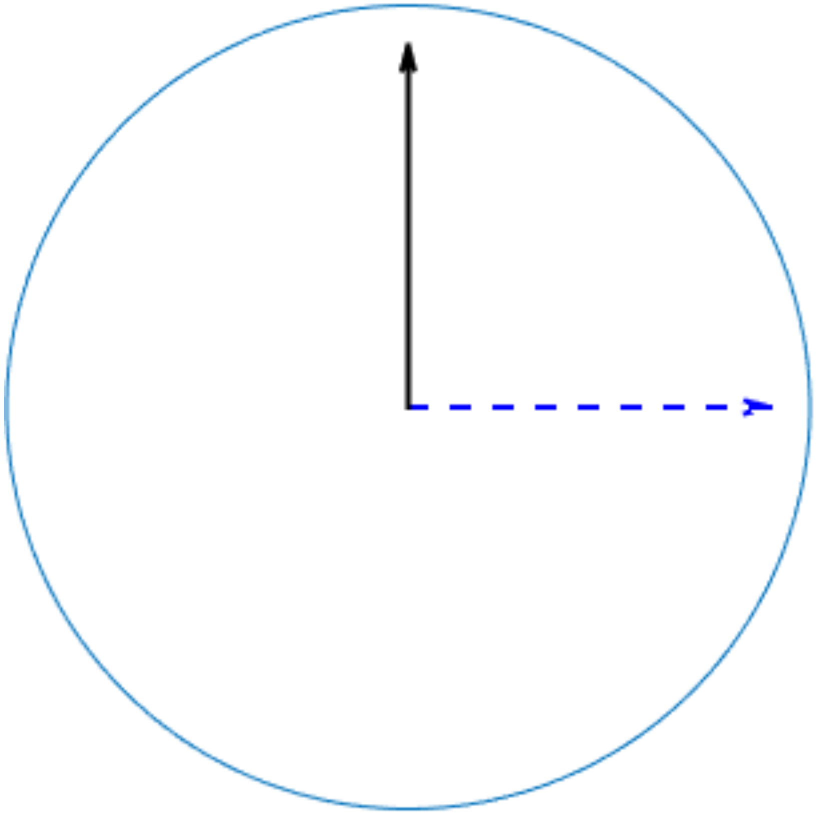

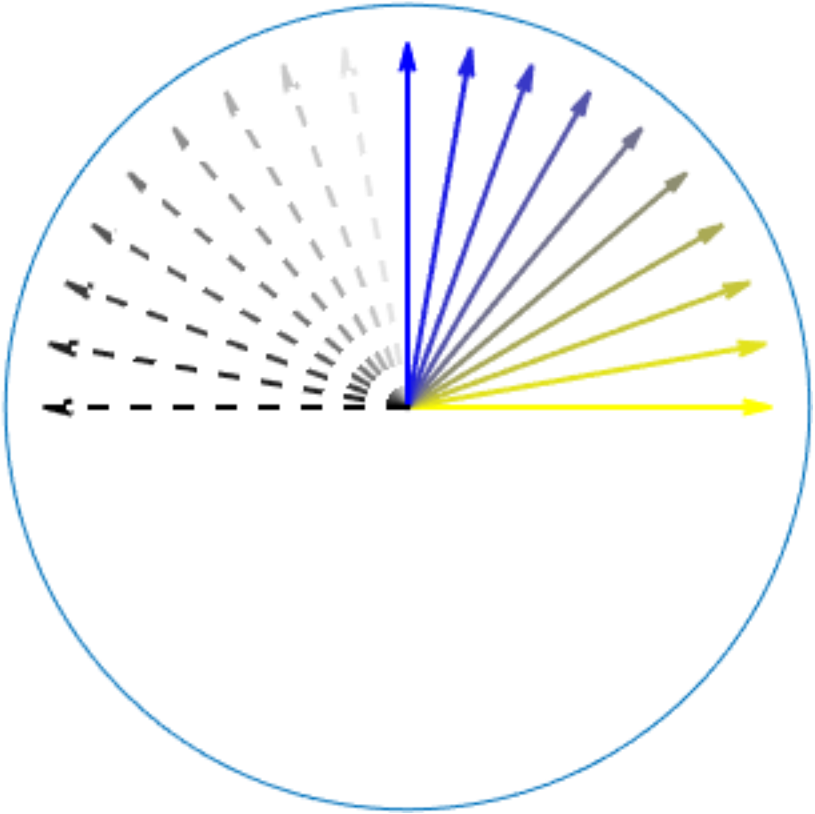

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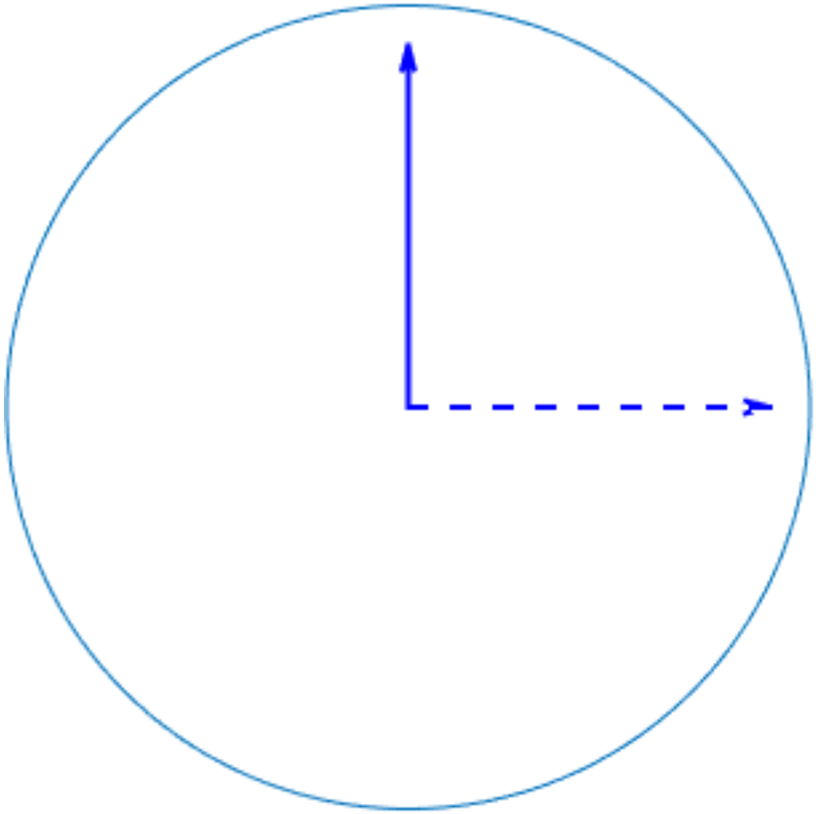

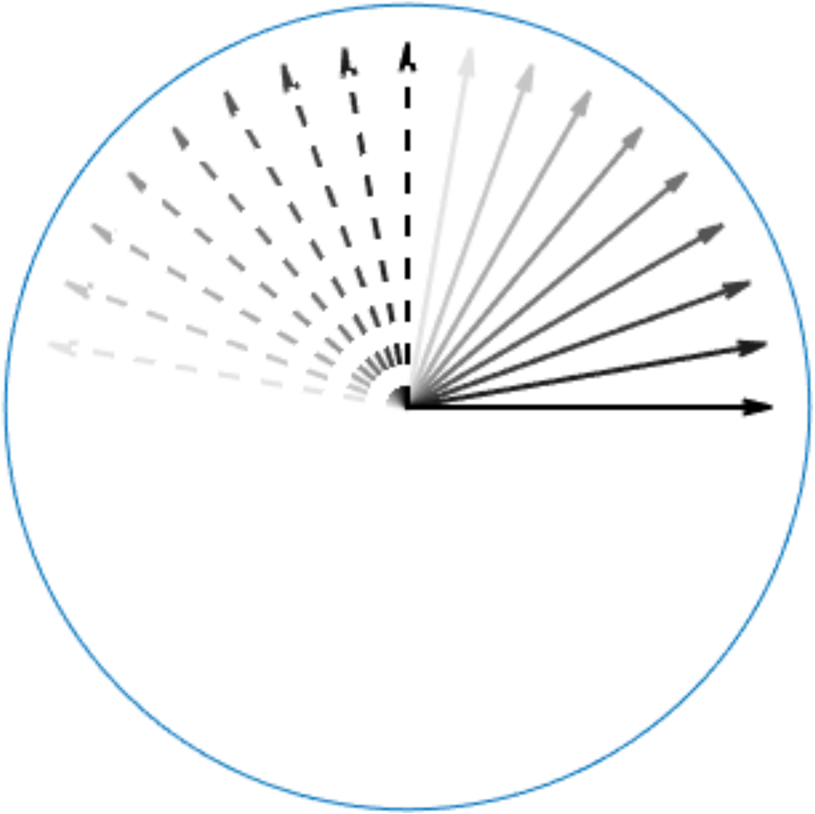

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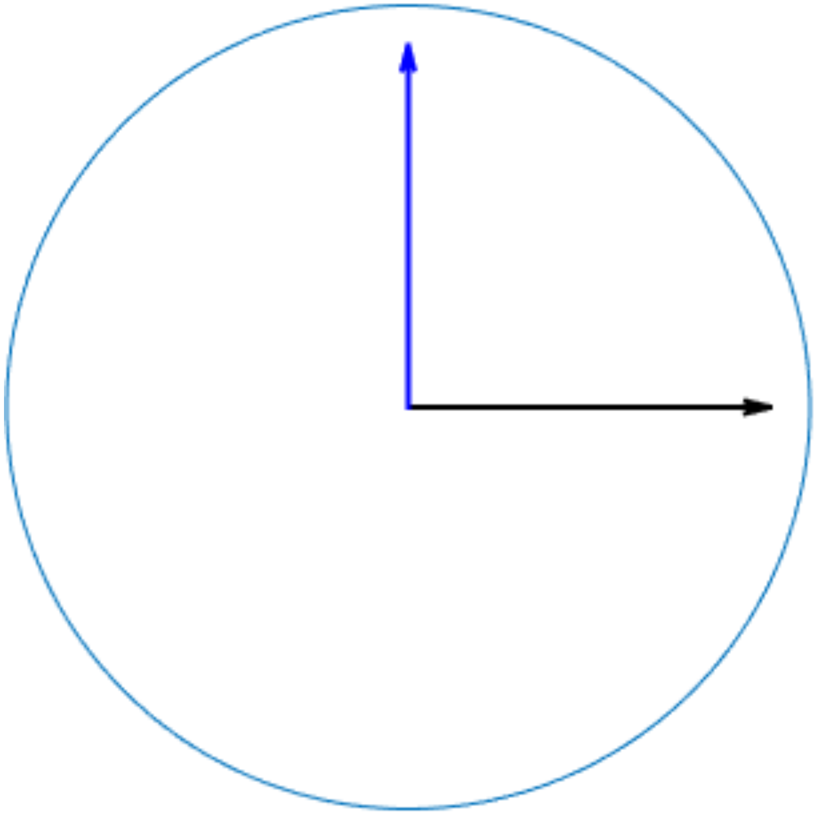
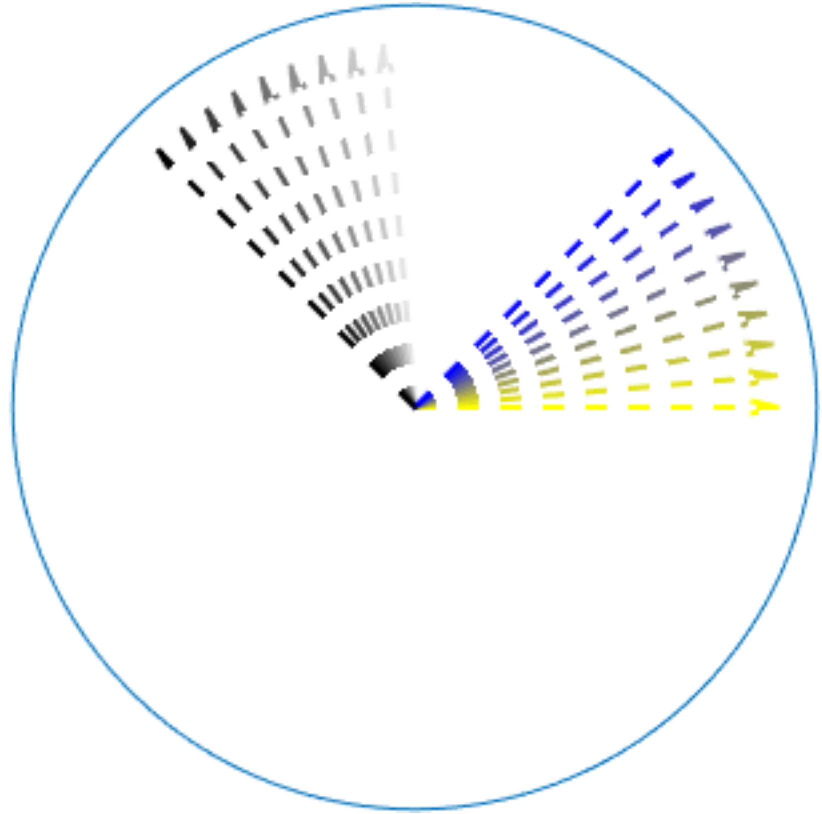




Basic Question: Are FUNTF Varieties Irreducible?

Proposition. *Suppose V is an algebraic variety such that*

(i) *the set of non-singular points of V is path-connected, and*

(ii) *the set of non-singular points is dense in V.*

Then V is an irreducible algebraic variety.

Nonsingular Points are Dense

Proposition. *A frame $F \in \mathcal{F}_{3,6}$ is orthodecomposable if and only if there are two distinct frame vectors f_i and f_j that are parallel.*

Proof. If F is orthodecomposable, then there is a partition of F into F_1 and F_2 so that the linear spans of the vectors in F_1 and F_2 (denoted V_1 and V_2) are non-trivial orthogonal subspaces of $\mathbb{R}^3$. Consequently, either V_1 or V_2 has dimension equal to 1, and hence by the tight frame bound condition either F_1 or F_2 consists of two parallel vectors.

On the other hand, assuming that there are vectors f_i and f_j which are parallel, the tight frame bound condition requires that all other vectors in F are orthogonal to f_i and f_j. Consequently, F is orthodecomposable. $\square$

Connectivity of the Non-singular FUNTFs

Proposition. *If $F \in \mathcal{F}_{3,6}$ is orthodecomposable, then it is a union of two orthonormal bases.*

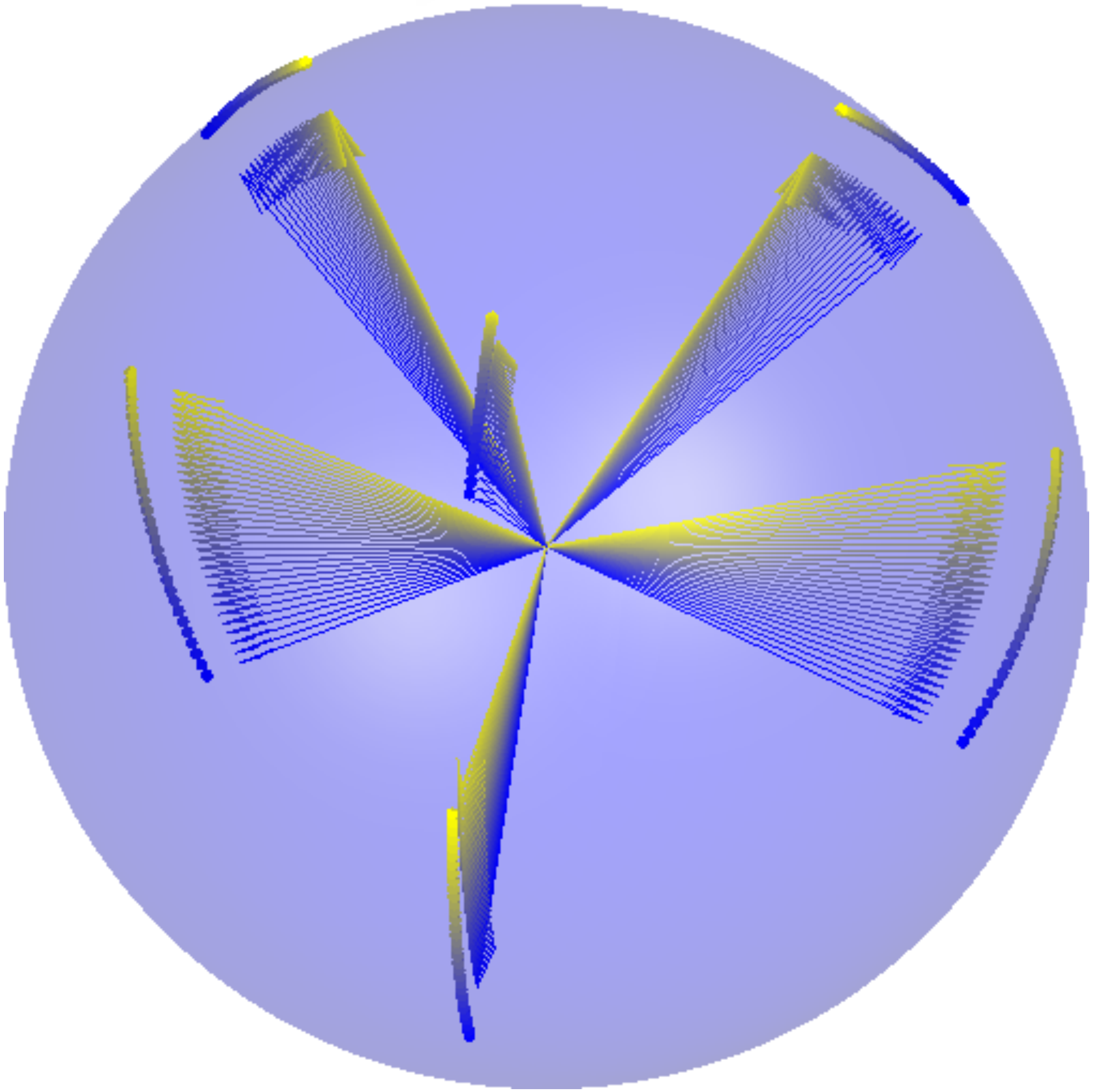
Proposition. *The non-singular points of $\mathcal{F}_{3,6}$ are dense in $\mathcal{F}_{3,6}$.*

Connectivity of the Non-singular FUNTFs

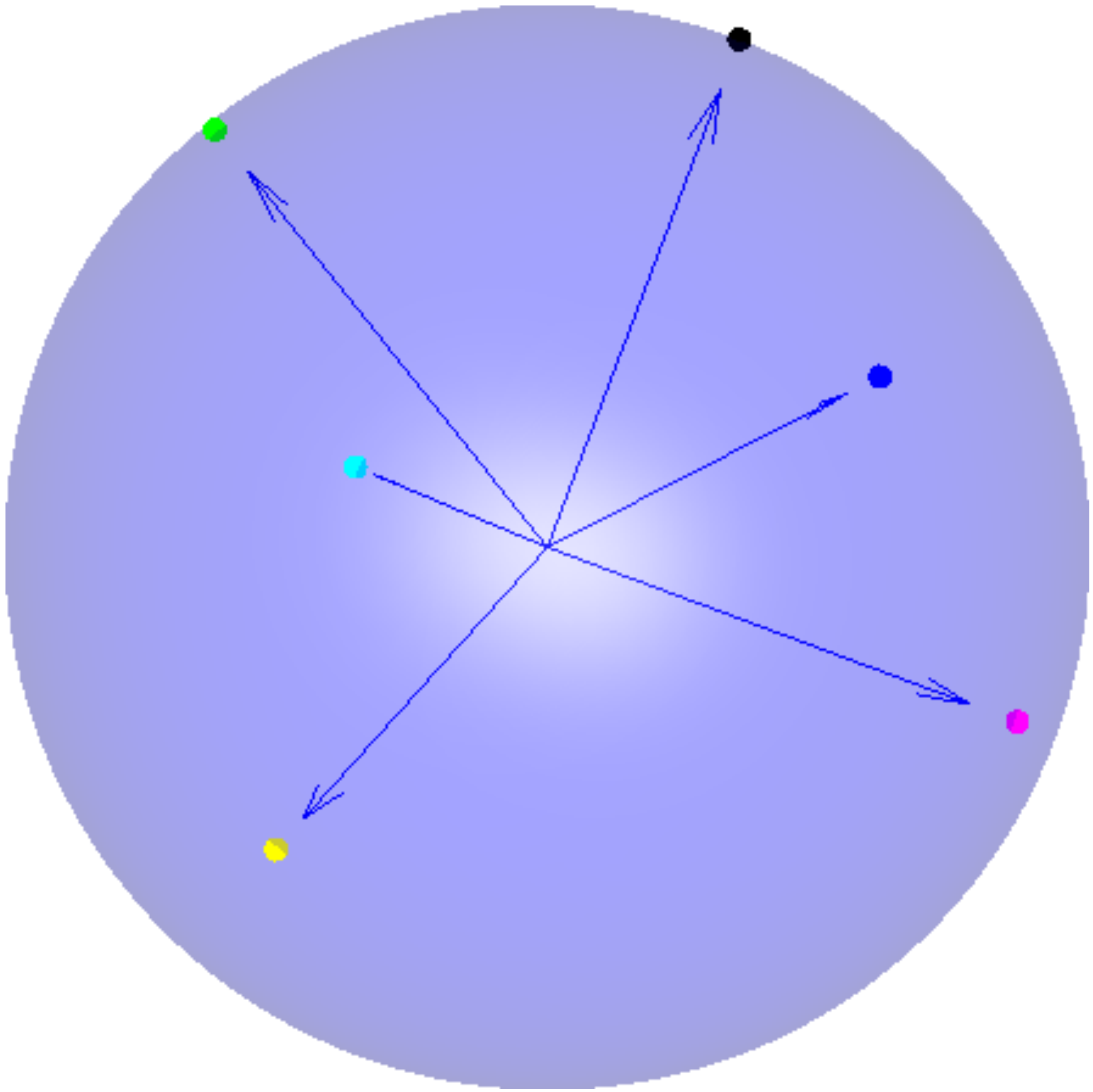
Connect to a frame with eigensteps

$$\begin{pmatrix} 1 & 3/2 & 3/2 & 2 & 2 & 2 \\ 0 & 1/2 & 3/2 & 3/2 & 2 & 2 \\ 0 & 0 & 0 & 1/2 & 1 & 2 \end{pmatrix}$$

Connectivity of the Non-singular FUNTFs



Connectivity of the Non-singular FUNTFs



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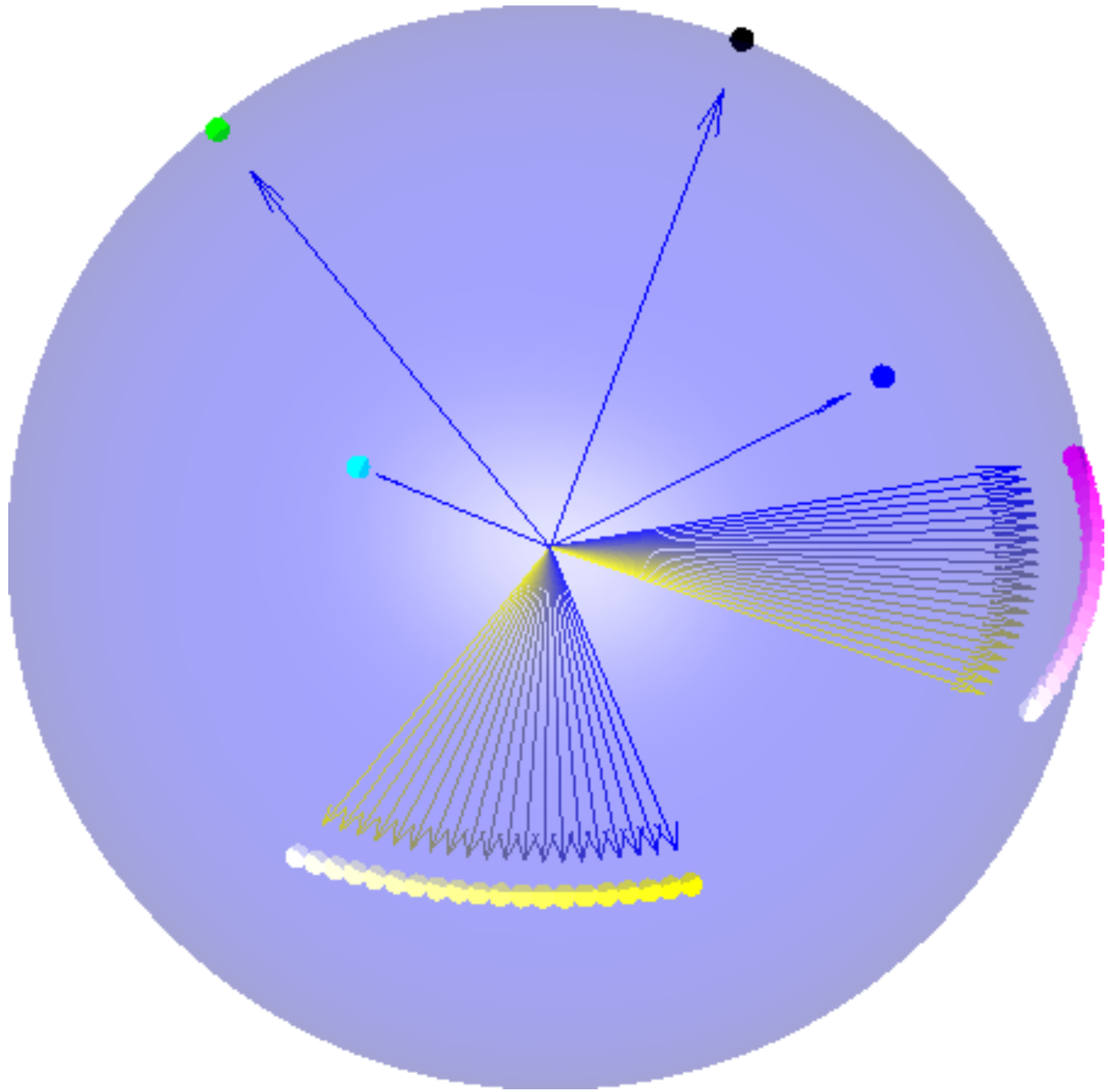

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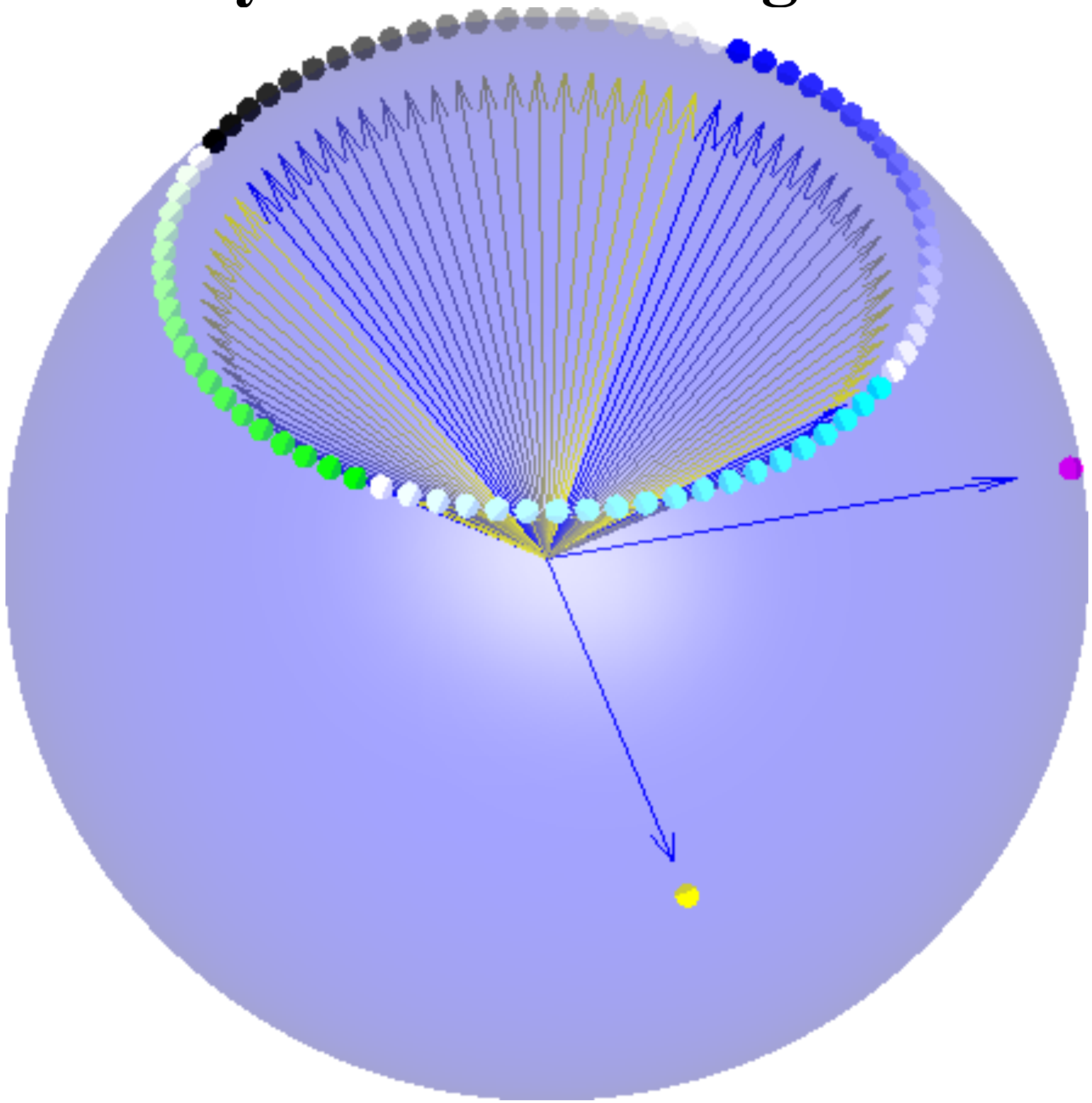

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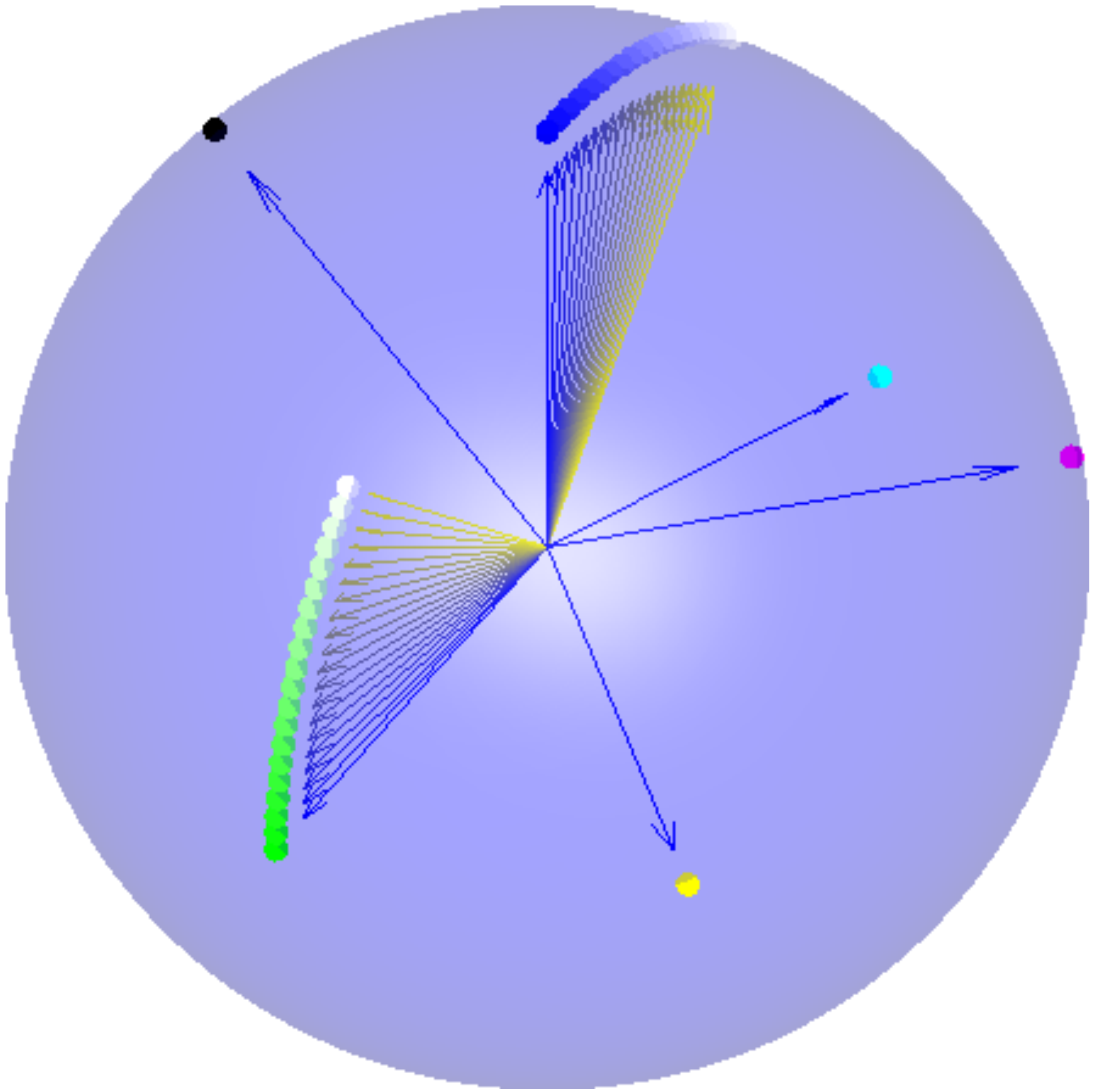

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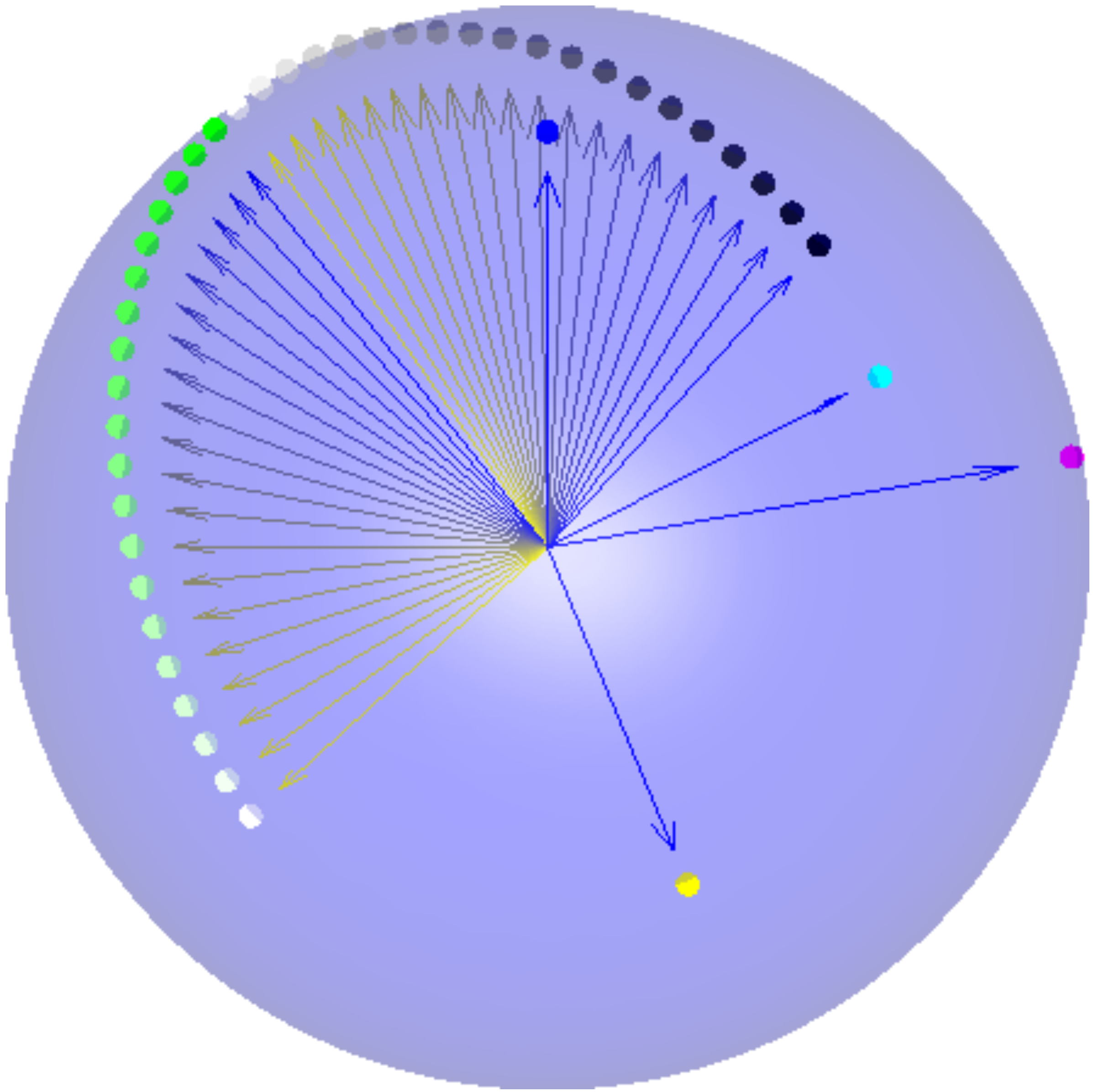

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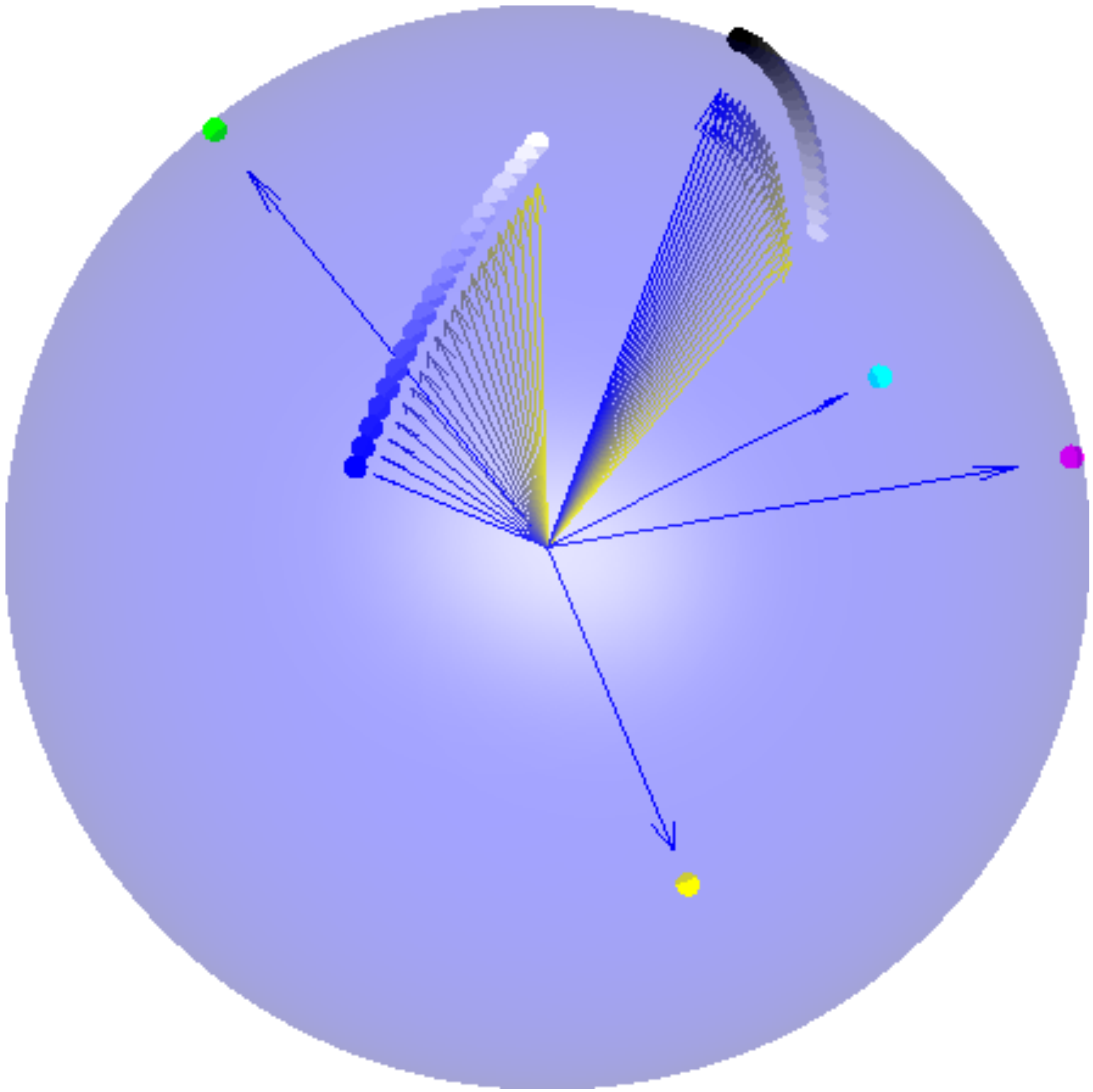

Open Problem: Describe the Local Geometry Around OD Frames

Open Problem: Calculate blowups of the FUNTF Varieties

Open Problem: Are there any FUNTF Varieties such that All the Members are Permutation Equivalent to Frames which Map to the Interior of the Eigensteps Polytope?

Open Problem: Compute the fundamental groups of the FUNTF Varieties

Open Problem: Construct the FUNTF varieties as configuration spaces

Idea

Set $\varphi(f) = ff^*$ and note that $\varphi(S)$ is diffeomorphic to $\mathbb{RP}^{d-1}$.

FUNTFs correspond to chains in $M_{d,d}$ from 0 to $\frac{N}{d} I_d$ with links in $\varphi(\mathcal{S}^{d-1})$.