Four open problems in frame theory

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Frames and Algebraic & Combinatorial Geometry

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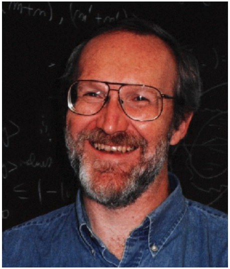
Part I

The Paulsen Problem



A quick recap of unit norm tight frames

Definition of $M \times N$ UNTF:

(i) Rows have norm $\sqrt{N/M}$

(ii) Rows are orthogonal

(iii) Columns have norm 1

UNTFs form an algebraic variety of known dimension

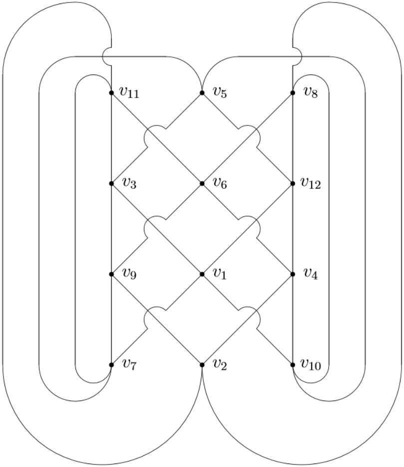
Singular points of the variety are the **orthodecomposable frames**

UNTF variety is a manifold when M and N are relatively prime

Strawn, J. Fourier Anal. Appl., 2011

A quick recap of unit norm tight frames

Topology of the real 2×4 UNTFs modulo rotation:



Dykema, Strawn, Internat. J. Pure Appl. Math., 2006

A quick recap of unit norm tight frames

To find a UNTF, pick a matrix and optimize for UNTF-ness

When can we be sure that a UNTF is close by?

The Paulsen Problem

If Φ is close to being a UNTF, that is,

$$\Phi\Phi^* \approx \tfrac{N}{M} I, \qquad \mathrm{diag}(\Phi^*\Phi) \approx \mathbf{1},$$

how far is the closest UNTF?

(One might pose the analogous question for ETFs...)

Bodmann, Casazza, J. Funct. Anal., 2010

Progress on the Paulsen problem

Easy solution: Apply the Łojasiewicz inequality to the function

$$f(\Phi) = \left\|\Phi\Phi^* - \tfrac{N}{M} I\right\|_{\mathrm{F}}^2 + \sum_{i=1}^{N} \left(\|\varphi_i\|^2 - 1\right)^2$$

Observe $\mathsf{UNTF} = \{\Phi : f(\Phi) = 0\}$

For every $\epsilon > 0$, there exist $\alpha = \alpha(\epsilon)$ and $C = C(\epsilon)$ such that

$$f(\Phi) \leq \epsilon \quad \Rightarrow \quad \mathrm{dist}(\Phi, \mathsf{UNTF})^\alpha \leq C \cdot f(\Phi)$$

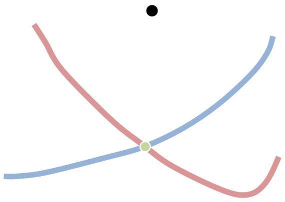
We want the smallest possible α in terms of M and N

Fernando, Gamboa, Math. Res. Lett., 2010

Progress on the Paulsen problem

Goal: Find nearby point in

$$\mathbf{UNTF} = \mathbf{TF} \cap \mathbf{UNF}$$



Progress on the Paulsen problem

Goal: Find nearby point in

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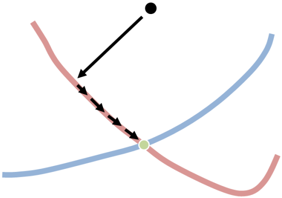


Method 1:

1. Project onto TF: $\Phi \mapsto (\frac{M}{N}\Phi\Phi^*)^{-1/2}\Phi$
2. Follow an ODE to flow through TF towards UNTF

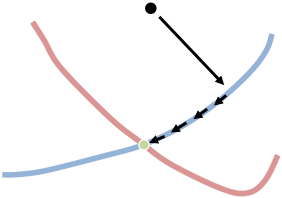
Result: $\alpha = 2$ when $\gcd(M, N) = 1$, no estimate otherwise

Bodmann, Casazza, J. Funct. Anal., 2010

Progress on the Paulsen problem

Goal: Find nearby point in

UNTF $=$ **TF** $\cap$ **UNF**



Method 2:

1. Project onto UNF: $\Phi \mapsto \Phi\,\mathrm{diag}(\Phi^*\Phi)^{-1/2}$
2. Locally minimize the frame potential in UNF
3. "Jump" when nearly orthodecomposable

Result: $\alpha = 2$ when $\gcd(M, N) = 1$, otherwise $\alpha \leq 2 \cdot 7^M$

Casazza, Fickus, M., Appl. Comput. Harmon. Anal., 2012

Better results with new techniques?

Definition of $M \times (N+1)$ **eigensteps matrix** $(\lambda_{ij})_{i=1,\ j=0}^{M\ \ \ N}$:

(i) 0th column is all zeros

(ii) Adjacent columns interlace:

$$\lambda_{M,j} \leq \lambda_{M,j+1} \leq \lambda_{M-1,j} \leq \cdots \leq \lambda_{2,j+1} \leq \lambda_{1,j} \leq \lambda_{1,j+1}$$

Facts:

- $E_{M,N} = \{M \times (N+1) \text{ eigensteps matrices}\}$ is a convex polytope
- $\Lambda\colon \mathbb{C}^{M \times N} \to E_{M,N}$, $\Lambda_{ij}(\Phi) = \lambda_i(\Phi_j \Phi_j^*)$ is onto, continuous
- $\Lambda(\mathrm{TF})$, $\Lambda(\mathrm{UNF})$, $\Lambda(\mathrm{UNTF})$ are convex subpolytopes of $E_{M,N}$

Open problem:

How does distance in $\mathbb{C}^{M \times N}$ relate to distance in $E_{M,N}$?

Cahill, Fickus, M., Poteet, Strawn, Appl. Comput. Harmon. Anal., 2013

Part II

The Fickus Conjecture

A quick recap of equiangular tight frames

Goal: Find optimal packings of lines through the origin

Solution: prove uniform bound, then achieve equality in bound

1. Welch bound: $\max_{i \neq j} |\langle \varphi_i, \varphi_j \rangle| \geq \sqrt{\frac{N-M}{M(N-1)}}$

2. Φ achieves equality in Welch bound iff Φ is an ETF

Strohmer, Heath, Appl. Comput. Harmon. Anal., 2003

A quick recap of equiangular tight frames

Definition of $M \times N$ ETF:

(i) Rows have norm $\sqrt{N/M}$

(ii) Rows are orthogonal

(iii) Columns have norm 1

(iv) Columns satisfy $|\langle \varphi_i, \varphi_j \rangle| = \sqrt{\frac{N-M}{M(N-1)}}$ whenever $i \neq j$

In the real case, we know a lot:

- $\Phi^*\Phi \longleftrightarrow$ adjacency matrix of strongly regular graph
- ETF exists only if $f(M,N)$ is integer/nonnegative for several f

Open problem: Necessary/sufficient conditions for complex ETFs

Sustik, Tropp, Dhillon, Heath, Linear Algebra Appl., 2007

Waldron, Linear Algebra Appl., 2009

A quick recap of equiangular tight frames

After investigating all known ETFs, Matt posed a conjecture:

The Fickus Conjecture

Consider the three quantities:

$$M, \qquad N - M, \qquad N - 1.$$

An $M \times N$ ETF exists only if one of these quantities divides the product of the other two.

- Prove it, then Matt owes you US$200
- Disprove it, then Matt owes you US$100

Fickus, M., arXiv:1504.00253

M., dustingmixon.wordpress.com/2015/07/08/conjectures-from-sampta/

Towards a proof of the Fickus Conjecture

Our knowledge to date: A complex ETF exists only if

- $N \in \{M, M+1\} \cup \left[M + \Omega(\sqrt{M}), M^2\right]$
- $(M, N) \neq (3, 8)$

How to prove the second condition:

1. Characterize ETFs with 667 polynomials in 12 variables
2. Take about an hour to compute a Gröbner basis
3. Find 1 in the ideal generated by the Gröbner basis
4. Conclude that no solutions exist

What's the next thing to try?

Strohmer, Heath, Appl. Comput. Harmon. Anal., 2003

Szöllősi, `arXiv:1402.6429`

Towards a proof of the Fickus Conjecture

Proposed program to prove that no $M \times N$ ETFs exist:

1. Find a nonnegative polynomial $p \in \mathbb{R}[x_1, \ldots, x_{2MN}]$ whose roots are the $M \times N$ ETFs
2. Show that $\min_x p(x) > 0$ (no roots means no ETFs)

For step 1, here's a choice for p:

$$p(\Phi) = \Big\| |\Phi^*\Phi|^2 - W \Big\|_{\mathrm{F}}^2, \quad W_{ii} = 1, \quad W_{ij} = \tfrac{N-M}{M(N-1)}$$

For step 2, exploit duality:

$$\min_x p(x) = \max_{p-\epsilon \geq 0} \epsilon$$

Unfortunately, testing for nonnegativity is NP-hard in general

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Parrilo, Math. Program., Ser. B, 2003

Towards a proof of the Fickus Conjecture

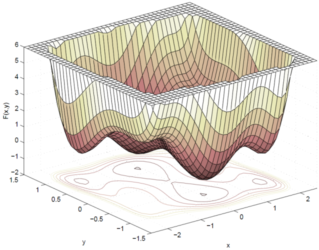
A **sum-of-squares** polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ has the form

$$f(x) = \sum_{i=1}^{k} g_i(x)^2, \qquad g_1, \ldots, g_k \in \mathbb{R}[x_1, \ldots, x_n]$$

- SOS is a convex subcone of nonnegative polynomials, and so

$$\max_{p-\epsilon \geq 0} \epsilon \geq \max_{p-\epsilon \in \mathrm{SOS}} \epsilon$$

- Bound is often tight, e.g., $p = F$
- RHS solved in polynomial time using semidefinite programming



$F(x, y) = 4x^2 - \frac{21}{10}x^4 + \frac{1}{3}x^6 + xy - 4y^2 + 4y^4$

Parrilo, Math. Program., Ser. B, 2003

Towards a proof of the Fickus Conjecture

Proposal: Find $\epsilon > 0$ such that $p - \epsilon$ is SOS

Good news: $p(\Phi) = \big\| |\Phi^*\Phi|^2 - W \big\|_{\mathrm{F}}^2$ is SOS

Bad news: Naïve SDP has $O((MN)^{16})$ matrix entries

Ought to exploit p's structure:

- p is sparse $\Rightarrow$ exponents in each g_i lie in a small known set
- p enjoys symmetries: $p(U\Phi) = p(\Phi)$, $p(\Phi P) = p(\Phi)$

Can we recover $(M, N) \neq (3, 8)$? Is $(M, N) \neq (4, 8)$ necessary?

Reznick, Duke Math. J., 1978

Gatermann, Parrilo, J. Pure Appl. Algebra, 2004

Part III

Zauner's Conjecture

Maximal ETFs

An $M \times N$ ETF exists only if $N \leq M^2$

An $M \times M^2$ ETF is called **maximal** or a **SIC-POVM**

Maximal ETF constructions are known for each

$$M \in \{1, 2, \ldots, 17, 19, 24, 28, 35, 48\}$$

Numerical evidence suggests existence whenever $M \leq 67$

Zauner, `gerhardzauner.at/sicfiducialsd.html`

Chein, Ph.D. thesis, U. Auckland, 2015

Scott, Grassl, J. Math. Physics, 2010

Maximal ETFs

Zauner's Conjecture

For each $M \geq 1$, there exists an $M \times M^2$ ETF (with very specific structure).

How to construct a maximal ETF:

1. Pick a function $\varphi\colon \mathbb{Z}/M\mathbb{Z} \to \mathbb{C}$ (the **fiducial** vector)
2. Take the **Gabor frame** $\Phi = \{T^a E^b \varphi\}_{a,b \in \mathbb{Z}/M\mathbb{Z}}$, where

$$(T^a\psi)(x) = \psi(x-a), \qquad (M^b\psi)(x) = e^{2\pi ibx/M}\psi(x)$$

3. Pray that Φ is an ETF

To date, if we have an $M \times M^2$ ETF, we have one that's Gabor

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Zauner, Ph.D. thesis, U. Vienna, 1999

Recent progress on Zauner's conjecture

Chein's program to find explicit maximal ETFs:

1. Take a numerically approximated ETF fiducial vector
2. Locally optimize to obtain many (say, 2000) digits of precision
3. Apply field structure conjectures to guess analytic expression
4. Verify ETF properties by symbolic computation

This program recently produced the first explicit 17×17^2 ETF

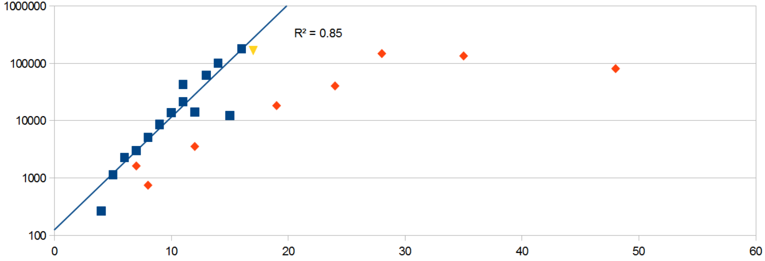
Conditionally finite-time algorithm! But step 3 is slow...

Chein, Ph.D. thesis, U. Auckland, 2015

Recent progress on Zauner's conjecture

Claim: Every method that reports explicit fiducial vectors is slow

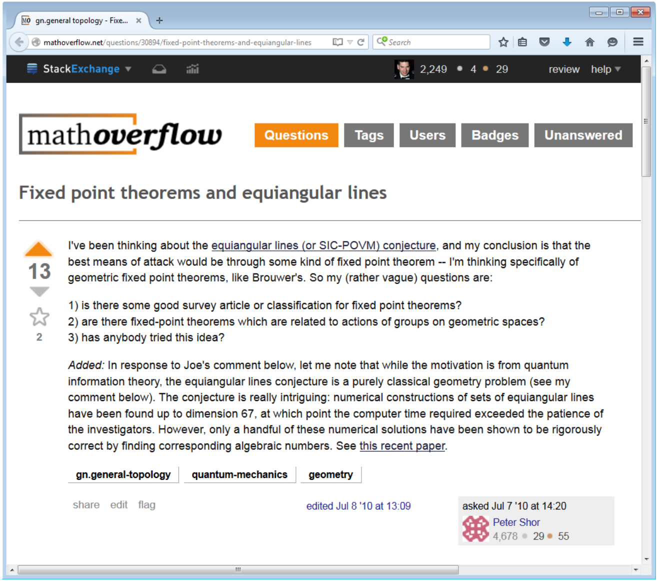
- Explicit fiducial vectors are available on Zauner's webpage
- Count characters in each fiducial vector's description and plot:



Exponential description length! We need an alternative...

M., dustingmixon.wordpress.com/2015/03/10/zauners-conjecture-is-true-in-dimension-17/

How to avoid being explicit?



Shor, `mathoverflow.net/questions/30894/fixed-point-theorems-and-equiangular-lines`

How to avoid being explicit?

Perhaps fiducial vectors are simpler in lifted space

For M odd, define the **discrete Wigner transform** by

$$(Wf)(t,\omega) = \frac{1}{\sqrt{M}} \sum_{\tau \in \mathbb{Z}/M\mathbb{Z}} f(t+\tfrac{\tau}{2})\overline{f(t-\tfrac{\tau}{2})}e^{-2\pi i \tau\omega/M}$$

Useful properties:

- $(Wf)(t,\omega) \in \mathbb{R}$ for every $t,\omega \in \mathbb{Z}/M\mathbb{Z}$
- $\langle Wf, Wg\rangle = |\langle f, g\rangle|^2$
- $W(T^a E^b \varphi) = T^{(a,b)}(W\varphi)$

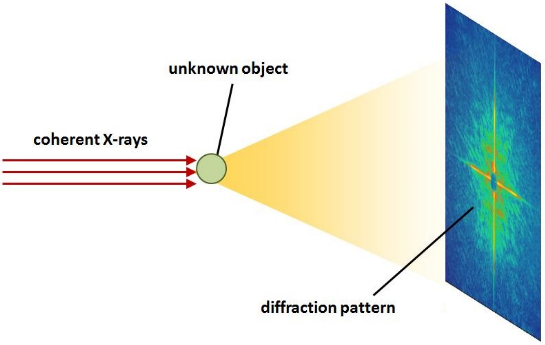
Goal: Find $F \in \mathrm{im}(W)$ such that translates of F are equiangular

Part IV

Vinzant's Conjecture

A quick intro to coherent diffractive imaging

Disclaimer: I am not a physicist.



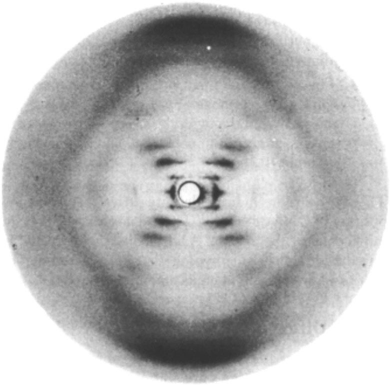
What does the diffraction pattern say about the object?

A quick intro to coherent diffractive imaging

Diffraction pattern can shed light on nanoscale structures:

- 1962 Nobel Prize
  (Watson, Crick, Wilkins)
  Deduced DNA's double helix structure

- 1985 Nobel Prize
  (Hauptman, Karle)
  Ad hoc "shake-and-bake" algorithm determined structures of small proteins and antibiotics



Watson, Crick, Nature, 1953

Hauptman, Karle, Am. Crystallogr. Assoc., 1953

A quick intro to coherent diffractive imaging

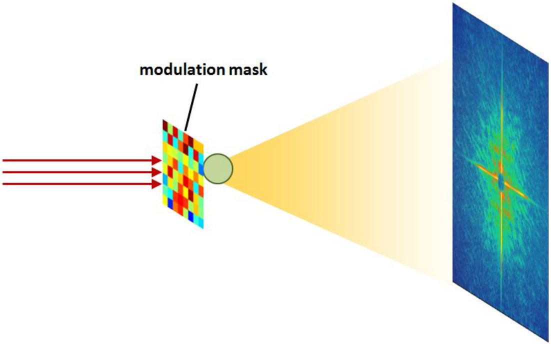
Modern goal: Find a way to systematically ~~win Nobel Prizes~~ recover the object x from its diffraction pattern $|Fx|^2$

The phase retrieval step is severely underdetermined, so more information is necessary:

- A priori knowledge about object
- Additional measurements

A quick intro to coherent diffractive imaging

Modulate the X-rays to change the object's appearance



Claim: If chosen properly, masks $\{\mu_r\}_{r=1}^R$ give complete info

To solve: $|\Phi^* x|^2 \mapsto x \bmod \mathbb{T}$, where $\Phi^* = [F\mu_1; F\mu_2; \ldots; F\mu_R]$

Candès, Eldar, Strohmer, Voroninski, SIAM J. Imaging Sci., 2013

The phase retrieval problem

Relax: Let $\Phi \in \mathbb{C}^{M \times N}$ be arbitrary

Goal: Recover any x up to global phase from $|\Phi^* x|^2$

How large must N be relative to M?

The $4M-4$ Conjecture

(a) If $N < 4M-4$, then $(x \bmod \mathbb{T}) \mapsto |\Phi^* x|^2$ is not injective.

(b) If $N \geq 4M-4$, then $(x \bmod \mathbb{T}) \mapsto |\Phi^* x|^2$ is injective for generic Φ.

Bandeira, Cahill, M., Nelson, Appl. Comput. Harmon. Anal., 2014

The phase retrieval problem

What we now know:

- Part (a) holds for whenever $M = 2^k + 1$
- Part (b) holds for all M

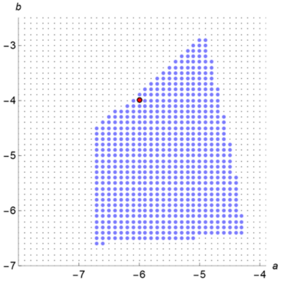
Conca, Edidin, Hering, Vinzant, Appl. Comput. Harmon. Anal., 2015

Vinzant, SampTA, 2015

The phase retrieval problem

What we now know:

- Part (a) holds for whenever $M = 2^k + 1$
- Part (b) holds for all M
- Part (a) **does not hold** for $M = 4$ (!)



Conca, Edidin, Hering, Vinzant, Appl. Comput. Harmon. Anal., 2015

Vinzant, SampTA, 2015

The phase retrieval problem

Observe that injectivity is a property of $\mathrm{im}(\Phi^*)$

Vinzant's Conjecture

Draw $\mathrm{im}(\Phi^*)$ uniformly from Grassmannian of M-dim subspaces of $\mathbb{C}^{4M-5}$.
Let p_M denote the probability that $(x \bmod \mathbb{T}) \mapsto |\Phi^* x|^2$ is injective.

(a) $p_M < 1$ for all M.

(b) $\lim_{M\to\infty} p_M = 0$.

- Prove part (a), then Cynthia owes you a can of Coca-Cola
- Prove part (b), then Cynthia owes you US$100

Summary

- The Paulsen Problem *eigensteps isometry?*
- The Fickus Conjecture *SOS programming?*
- Zauner's Conjecture *implicit fiducial vectors?*
- Vinzant's Conjecture *???*

Questions?

Google **short fat matrices** to find more on my research blog