Full Spark Gabor Frames in Finite Dimensions

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Cyclic shift operator

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Modulation operator ($\omega = \exp(2\pi i/N)$)

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Definition

A Gabor frame with window $\varphi \in \mathbb{C}^N$ is the set of all time-frequency translates of φ:

$$M^\lambda T^\kappa \varphi,\ 0 \leq \kappa, \lambda \leq N-1,$$

also called a Weyl-Heisenberg orbit.

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Problem (Lawrence, Pfander, Walnut, 2005)

Are there $\varphi \in \mathbb{C}^N$ such that $(\varphi, \mathbb{Z}_N^2)$ has full spark, i. e. (φ, Λ) is linearly independent for all $\Lambda \subseteq \mathbb{Z}_N^2$ with $|\Lambda| = N$?

Progress

- N prime, LPW, 2005.
- $N = 4, 6$, Krahmer, Pfander, Rashkov, 2008.
- $N = 8$, Appleby, Bengtsson, Blanchfield, Dang, 2013.
- $N \in \mathbb{N}$, M, 2013.

Applications

- Signal recovery
- Operator identification and sampling
- Compressive sensing

Signal Recovery

$(\varphi, \mathbb{Z}_N^2) = \{\varphi_k\}$ is an *equal norm tight frame*: we have $\|\varphi_k\| = \|\varphi\|$ and

$$\sum_k |\langle f, \varphi_k \rangle|^2 = N\|\varphi\|^2 \|f\|^2.$$

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$$f = \sum_{k \in K} \langle f, \varphi_k \rangle \tilde{\varphi}_k,$$

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Compressive Sensing

Problem

Reconstruct a vector that is a linear combination of at most s elements of the form $M_\xi T_x \varphi$ (sparse signal).

Identify $\mathcal{H}_s$, which is the set of all operators that are obtained as linear combinations of at most s time-frequency operators, $M_\xi T_x$.

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So, if $f = H\varphi$ for some $H \in \mathcal{H}_s$, can we recover a representation of f with respect to $(\varphi, G \times \hat{G})$, such that the number of nonzero coefficients is as small as possible (smaller than s)?

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SIC-POVM

Problem

Are there $\varphi_1, \ldots, \varphi_{N^2} \in \mathbb{C}^N$ *such that* $\|\varphi_i\| = 1$ *and* $|\langle \varphi_i, \varphi_j \rangle| = \frac{1}{\sqrt{N+1}}$ *for all* $i \neq j$*?*

Zauner ('99) conjectured that there is $\varphi \in \mathbb{C}^N$ such that $(\varphi, \mathbb{Z}_N^2)$ is a solution. Verified numerically for $N \leq 67$. Such a set is also a complex 2-design.

The main idea

Let $\Lambda \subseteq \mathbb{Z}_N^2$ with $|\Lambda| = N$. The column vectors $M^\lambda T^\kappa z$ form a matrix, denoted by D_Λ, where

$$z = (z_0, z_1, \dots, z_{N-1}) \in \mathbb{C}^N$$

is a variable vector. Define

$$P_\Lambda(z) = \det(D_\Lambda).$$

P_Λ is a homogeneous polynomial in N complex variables, of degree N. The set of zeroes of P_Λ is either the entire space $\mathbb{C}^N$ or has measure zero.

$(z, \mathbb{Z}_N^2)$ for $N = 2, 3$.

$$\left(\begin{array}{cc|cc} z_0 & z_0 & z_1 & z_1 \\ z_1 & -z_1 & z_0 & -z_0 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc|ccc} z_0 & z_0 & z_0 & z_2 & z_2 & z_2 & z_1 & z_1 & z_1 \\ z_1 & \omega z_1 & \omega^2 z_1 & z_0 & \omega z_0 & \omega^2 z_0 & z_2 & \omega z_2 & \omega^2 z_2 \\ z_2 & \omega^2 z_2 & \omega z_2 & z_1 & \omega^2 z_1 & \omega z_1 & z_0 & \omega^2 z_0 & \omega z_0 \end{array}\right)$$

Examples

Let $\Lambda = \{(0,1),(1,0)\} \subseteq \mathbb{Z}_2^2$.

$$D_\Lambda = \begin{pmatrix} z_0 & z_1 \\ -z_1 & z_0 \end{pmatrix}, \quad P_\Lambda = z_0^2 + z_1^2$$

Examples

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$$D_\Lambda = \begin{pmatrix} z_0 & z_0 & z_2 \\ z_1 & \omega z_1 & \omega z_0 \\ z_2 & \omega^2 z_2 & \omega^2 z_1 \end{pmatrix}$$

$$\begin{aligned} P_\Lambda &= z_0 z_1^2 + \omega z_0^2 z_2 + \omega^2 z_1 z_2^2 - z_0^2 z_2 - \omega^2 z_0 z_1^2 - \omega z_1 z_2^2 \\ &= (1-\omega^2) z_0 z_1^2 + (\omega - 1) z_0^2 z_2 + (\omega^2 - \omega) z_1 z_2^2 \end{aligned}$$

Let $\Lambda = \{(0,0),(0,2),(0,3),(4,1),(4,5),(5,0)\} \subseteq \mathbb{Z}_6^2$. The columns are $z, M^2z, M^3z, MT^4z, M^5T^4z, T^5z$

$$D_\Lambda = \left(\underbrace{\begin{matrix} z_0 & z_0 & z_0 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 \\ z_2 & \omega^4 z_2 & z_2 \\ z_3 & z_3 & \omega^3 z_3 \\ z_4 & \omega^2 z_4 & z_4 \\ z_5 & \omega^4 z_5 & \omega^3 z_5 \end{matrix}}_{D_0} \quad \underbrace{\begin{matrix} z_2 & z_2 \\ \omega z_3 & \omega^5 z_3 \\ \omega^2 z_4 & \omega^4 z_4 \\ \omega^3 z_5 & \omega^3 z_5 \\ \omega^4 z_0 & \omega^2 z_0 \\ \omega^5 z_1 & \omega z_1 \end{matrix}}_{D_4} \quad \underbrace{\begin{matrix} z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \\ z_0 \end{matrix}}_{D_5}\right)$$

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Every diagonal gives the same monomial: here, it is $z_0^3 z_1 z_2 z_5$.

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Diagonal Union of Blocks (DUB)

Definition

Write $D_\Lambda = (D_0|D_1|\cdots|D_{N-1})$, where the columns of D_i have z_0 in the ith row. If D_i is a $N \times l_i$ matrix, then a *DUB* is a union of square submatrices $B_0, \ldots, B_{N-1}$ containing a diagonal, such that B_i is a $l_i \times l_i$ submatrix of D_i.

For $\sigma = \iota$ we have

$$c_\iota Z^\iota = \begin{vmatrix} z_0 & z_0 & z_0 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 \\ z_2 & \omega^4 z_2 & z_2 \end{vmatrix} \cdot \begin{vmatrix} \omega^3 z_5 & \omega^3 z_5 \\ \omega^4 z_0 & \omega^2 z_0 \end{vmatrix} \cdot |z_0|,$$

so

$$c_\iota = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega^3 \\ 1 & \omega^4 & 1 \end{vmatrix} \cdot \begin{vmatrix} \omega^3 & \omega^3 \\ \omega^4 & \omega^2 \end{vmatrix} \cdot |1| \neq 0$$

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$$D_\Lambda = \begin{pmatrix} z_0 & z_0 & z_0 & z_2 & z_2 & z_1 \\ z_1 & \omega^2 z_1 & \omega^3 z_1 & \omega z_3 & \omega^5 z_3 & z_2 \\ z_2 & \omega^4 z_2 & z_2 & \omega^2 z_4 & \omega^4 z_4 & z_3 \\ z_3 & z_3 & \omega^3 z_3 & \omega^3 z_5 & \omega^3 z_5 & z_4 \\ z_4 & \omega^2 z_4 & z_4 & \omega^4 z_0 & \omega^2 z_0 & z_5 \\ \underbrace{z_5 \quad \omega^4 z_5 \quad \omega^3 z_5}_{D_0} & & & \underbrace{\omega^5 z_1 \quad \omega z_1}_{D_4} & & \underbrace{z_0}_{D_5} \end{pmatrix}$$

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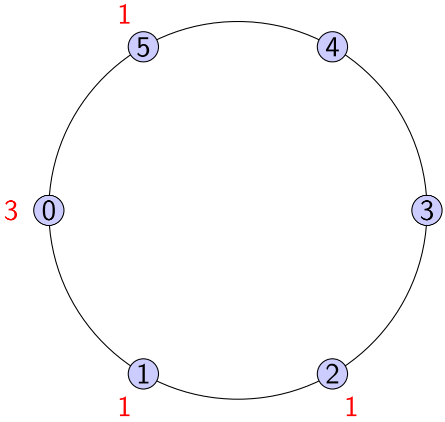
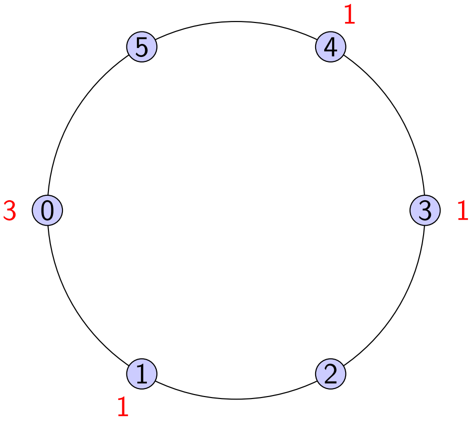
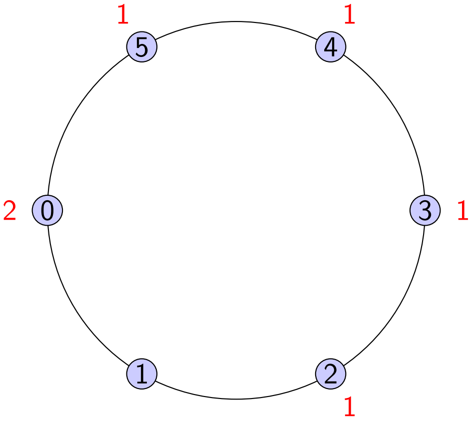
$\sigma = \iota$. Monomial: $z_0^3 z_1 z_2 z_5$.



$\sigma = (23)$. Monomial: $z_0^3 z_1 z_3 z_4$.



$\sigma = (14)$. Monomial: $z_0^2 z_2 z_3 z_4 z_5$.



Associate Z^σ to the random variable X_σ, as follows: if

$$Z^\sigma = z_0^{\alpha_0} z_1^{\alpha_1} \cdots z_{N-1}^{\alpha_{N-1}}$$

define

$$P[X_\sigma = i] = \frac{\alpha_i}{N}$$

Then, $E[X_\sigma^2]$ is minimized *uniquely*.

Theorem

In any dimension, in every D_Λ, the consecutive index monomial Z is obtained uniquely.

The concentration of indices of the CI monomial around a certain number is maximal.

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Let ξ be a transcendental number or an algebraic number whose degree over $\mathbb{Q}(\omega)$ is $> N(N-1)^2$. Then

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generates a full spark Gabor frame.

Corollary

Let $N \geq 4$ and ζ be any primitive root of unity, of order $(N-1)^4$. Then

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Thank you!