Frames in finite-dimensional spaces

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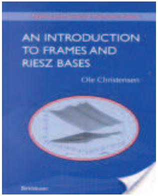
- An Introduction to frames and Riesz bases, Birkhäuser 2002.
- Second expanded edition (720 pages), Spring 2016



- Chapter 1: Frames in finite-dimensional spaces.
- If you want a pdf-file with Chapter 1 - contact me at ochr@dtu.dk

Plan for the talk

- Frames in finite-dimensional versus infinite-dimensional spaces;
- (Explicit constructions of tight frames in $\mathbb{C}^n$ with desirable properties)
  (Talks by Fickus, Mixon, Strawn)
- Tight frames versus dual pairs of frames in $\mathbb{C}^n$;
- Gabor frames in $L^2(\mathbb{R})$ and dual pairs;
- From Gabor frames in $L^2(\mathbb{R})$ to Gabor frames in $\mathbb{C}^n$ through sampling and periodization.
  (Talk by Malikiosis)
- 6 open problems along the way.

Key purpose of frame theory

Let V denote a vector space.

Want: Expansions

$$f = \sum c_k f_k$$

of signals $f \in V$ in terms of convenient building blocks f_k.

Desirable properties could be:

- Easy to calculate the coefficients c_k;
- Only few large coefficients c_k for the relevant signals f;
- Stability against noise or removal of elements.

The vector space can be

- A finite-dimensional vector space with inner product, typically $\mathbb{R}^n$ or $\mathbb{C}^n$;
- An infinite-dimensional Hilbert space; either an abstract space, or a concrete space, typically $L^2(\mathbb{R})$, $\ell^2(\mathbb{Z})$, or $L^2(0, L)$.
- A Banach space or a topological space ($L^p(\mathbb{R})$, Besov spaces, modulation spaces, Fréchet spaces)

Four classical tracks in frame theory

- Finite frames;
- Frame theory in separable Hilbert spaces;
- Gabor frames in $L^2(\mathbb{R})$;
- Wavelet frames in $L^2(\mathbb{R})$;
- (Geometric analysis: curvelets, shearlets,......)
- (Frames in Banach spaces, abstract generalizations, Hilbert C^* modules,.....).

To a large extent the 4 topics are developed independently of each other - but more coordination would be useful!

Frames - a generalization of orthonormal bases

Definition:

Let $\mathcal{H}$ denote a Hilbert space. A family of vectors $\{f_k\}_{k\in I}$ is a frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A||f||^2 \leq \sum_{k\in I} |\langle f, f_k\rangle|^2 \leq B||f||^2, \ \forall f \in \mathcal{H}.$$

The numbers A, B are called frame bounds.

The frame is tight if we can choose

$$A = B.$$

Note that

(i) If $\mathcal{H}$ is an infinite-dimensional Hilbert space, the index I must be infinite;

(ii) If $\mathcal{H}$ is finite-dimensional, the index set I can still be infinite (although in general not very natural)

General frame theory

Theorem Let $\{f_k\}_{k\in I}$ be a frame for $\mathcal{H}$. Then the following hold:

(i) The operator

$$S:\mathcal{H}\to\mathcal{H}, Sf := \sum_{k\in I}\langle f, f_k\rangle f_k$$

as well-defined, bounded, self-adjoint, and invertible;

(ii) Each $f\in\mathcal{H}$ has the expansion

$$f=\sum_{k\in I}\langle f, S^{-1}f_k\rangle f_k \quad \text{Tight case:} \quad f=\frac{1}{A}\sum_{k\in I}\langle f, f_k\rangle f_k$$

(iii) If $\{f_k\}_{k\in I}$ is a frame but not a basis, there exists families $\{g_k\}_{k\in I}\neq\{S^{-1}f_k\}_{k\in I}$ such that

$$f=\sum_{k\in I}\langle f, g_k\rangle f_k, \ \forall f\in\mathcal{H}.$$

Any such $\{g_k\}_{k=1}^{\infty}$ is called a dual frame.

Frames in finite-dimensional spaces

A frame for $\mathbb{C}^n$ is a collection of vectors $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ such that there exists constants $A, B > 0$ with the property

$$A||f||^2 \leq \sum_{k=1}^{m} |\langle f, f_k \rangle|^2 \leq B||f||^2, \ \forall f \in \mathbb{C}^n.$$

Proposition A family of vectors $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ is a frame if and only if

$$\mathrm{span}\{f_k\}_{k=1}^m = \mathbb{C}^n.$$

Corollary If $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ is a frame for $\mathbb{C}^n$, then $m \geq n$.

Frame theory in $\mathbb{C}^n$ is really “just” linear algebra!

Frames in finite-dimensional spaces

There are (at least) two tracks in frame theory in finite-dimensional spaces:

(i) Explicit construction of frames with desired properties;

(ii) Analysis of the interplay between frames in finite-dimensional spaces and in infinite-dimensional spaces.

The focus in this talk will be on (ii).

Bases and linear algebra

Classical results from linear algebra in $\mathbb{C}^n$

- Every set of linearly independent vectors $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ can be extended to a basis; i.e., there exist vectors $\{g_k\}_{k=1}^{\ell}$ such that

$$\{f_k\}_{k=1}^m \cup \{g_k\}_{k=1}^{\ell}$$

  is a basis for $\mathbb{C}^n$;
- Every family $\{f_k\}_{k=1}^m$ of vectors such that $\text{span}\{f_k\}_{k=1}^m = \mathbb{C}^n$, contains a basis; that is, there exists an index set I such that $\{f_k\}_{k\in\{1,\ldots,m\}\setminus I}$ is a basis for $\mathbb{C}^n$.

Frames in finite-dimensional spaces

Frame formulation:

Proposition:

(i) Every finite set of vectors $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ can be extended to a (tight) frame; i.e., there exist vectors $\{g_k\}_{k=1}^{\ell}$ such that

$$\{f_k\}_{k=1}^m \cup \{g_k\}_{k=1}^{\ell}$$

is a (tight) frame for $\mathbb{C}^n$;

(ii) Every frame $\{f_k\}_{k=1}^m$ for $\mathbb{C}^n$ contains a basis; that is, there exists an index set I such that $\{f_k\}_{k\in\{1,\dots,m\}\setminus I}$ is a basis for $\mathbb{C}^n$.

Frame theory in infinite-dimensional spaces is different:

Let $\mathcal{H}$ denote an infinite-dimensional separable Hilbert space.

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Every finite set of vectors in $\mathcal{H}$ can be extended to a tight frame.

Theorem (Casazza, C., 1995) There exist frames $\{f_k\}_{k=1}^{\infty}$, for which no subfamily $\{f_k\}_{k\in\mathbb{N}\setminus I}$ is a basis for $\mathcal{H}$.

Example Let $\{e_k\}_{k=1}^{\infty}$ denote an ONB for $\mathcal{H}$. Then the sequence

$$\{f_k\}_{k=1}^{\infty} := \left\{e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \cdots\right\}$$

is a tight frame; but no subfamily is a Riesz basis.

Frame theory in infinite-dimensional spaces is different:

A much more complicated result:

Proposition (Casazza, C., 1995) There exist tight frames $\{f_k\}_{k=1}^{\infty}$ with $||f_k|| = 1,\ \forall k \in \mathbb{N}$, for which no subfamily $\{f_k\}_{k\in\mathbb{N}\setminus I}$ is a basis for $\mathcal{H}$.

A sequence with a strange behavior

Example (C., 2001) Let $\{e_k\}_{k=1}^{\infty}$ denote an ONB for $\mathcal{H}$ and define $\{f_k\}_{k=1}^{\infty}$ by

$$f_k := e_k + e_{k+1}, \; k \in \mathbb{N}.$$

Then

(i) $\overline{\text{span}}\{f_k\}_{k=1}^{\infty} = \mathcal{H}$;

(ii) $\{f_k\}_{k=1}^{\infty}$ is a Bessel sequence, but not a frame;

(iii) There exists $f \in \mathcal{H}$ such that

$$f \neq \sum_{k=1}^{\infty} c_k f_k$$

for any choice of the coefficients c_k.

(iv) $\{f_k\}_{k=1}^{\infty}$ is minimal and its unique biorthogonal sequence $\{g_k\}_{k=1}^{\infty}$ is given by

$$g_k = (-1)^k \sum_{j=1}^{k} (-1)^j e_j, \; k \in \mathbb{N}.$$

A classical ONB for $\mathbb{C}^n$

Given $n \in \mathbb{N}$, let $\omega := e^{2\pi i/n}$ and consider the $n \times n$ discrete Fourier transform matrix (DFT) given by

$$\frac{1}{\sqrt{n}} \begin{pmatrix} 1 & 1 & 1 & \cdot & \cdot & 1 \\ 1 & \omega & \omega^2 & \cdot & \cdot & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdot & \cdot & \omega^{2(n-1)} \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdot & \cdot & \omega^{(n-1)(n-1)} \end{pmatrix}.$$

A classical ONB for $\mathbb{C}^n$

Given $n \in \mathbb{N}$, consider the n vectors $e_k,\ k = 1, \dots, n$ in $\mathbb{C}^n$, given by

$$e_k = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ e^{2\pi i \frac{k-1}{n}} \\ e^{4\pi i \frac{k-1}{n}} \\ \cdot \\ \cdot \\ e^{2\pi i (n-1) \frac{k-1}{n}} \end{pmatrix}, \ k = 1, \dots n.$$

Note that e_k is the kth column in the Fourier transform matrix (DFT).

Lemma: The vectors $\{e_k\}_{k=1}^n$ constitute an orthonormal basis for $\mathbb{C}^n$.

Tight frames in $\mathbb{C}^n$ - the first construction

Construction by Zimmermann (2001), motivated by question by Feichtinger:

Theorem: Let $m > n$ and define the vectors $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ by

$$f_k = \frac{1}{\sqrt{m}} \left(\begin{array}{c} 1 \\ e^{2\pi i \frac{k-1}{m}} \\ \cdot \\ \cdot \\ e^{2\pi i (n-1) \frac{k-1}{m}} \end{array} \right), \quad k = 1, 2, \dots, m.$$

Then $\{f_k\}_{k=1}^m$ is a tight overcomplete frame for $\mathbb{C}^n$ with frame bound equal to one, and $||f_k|| = \sqrt{\frac{n}{m}}$ for all k.

Note that the vectors f_k consist of the first n coordinates of the Fourier ONB for $\mathbb{C}^m$. The frame $\{f_k\}_{k=1}^m$ in $\mathbb{C}^n$ is called a harmonic frame.

Directions in frame theory in $\mathbb{C}^n$

- The result by Zimmermann can be seen as the starting point for the explosion in explicit construction of tight frames.
- Benedetto & Fickus (2003): Characterization of finite normalized tight frames using the frame potential.
- Casazza: papers with Leon (2006) & Leonhard (2008) on finite equal-norm frames.
- Casazza, Kovačević (2003): Equal-norm tight frames, erasures
- Benedetto, Powell, Yilmaz: Sigma-Delta quantization (2006), followed by a series of papers by Blum, Lammers, Powell, Yilmaz
- Strohmer (2003/2008): equiangular tight frames.
- Bodmann, Casazza, Kutyniok (2011): a quantitative notion for redundancy for finite frames.

Wish lish for frames in $\mathbb{C}^n$

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The issue of the length of the frame vectors is sometimes tricky!

Open problem posed by Thomas Strohmer, SAMPTA 2015

Let $\{f_k\}_{k=1}^m$ be a frame for $\mathbb{C}^n$, for which we only know the direction of the vectors f_k but not the norms $||f_k||$. Assume that we for an unknown vector $f \in \mathbb{C}^n$ know the inner products

$$\langle f, f_k \rangle,\ k = 1, \ldots, m.$$

How - and under which conditions - can we recover f?

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- The question is well-posed: since a frame is complete, knowledge of the numbers in $\langle f, f_k \rangle$ determines the vector f uniquely.
- If we actually know the norms $||f_k||$, we know the frame completely, and knowledge of the numbers $\langle f, f_k \rangle$ allow us to compute the frame operator

$$Sf = \sum_{k=1}^{m} \langle f, f_k \rangle f_k$$

and apply the frame decomposition

$$f = \sum^{m} \langle f, f_k \rangle S^{-1} f_k$$

Equiangular frames

If the elements in $\{f_k\}_{k=1}^m$ have the same length, the condition of being equiangular amounts to the existence of a constant C such that

$$|\langle f_k, f_j\rangle| = C, \ \forall k \neq j.$$

In particular, any orthonormal basis $\{e_k\}_{k=1}^n$ for $\mathbb{C}^n$ is equiangular.
Theorem (Strohmer & Heath, 2003) Consider a unit-norm frame $\{f_k\}_{k=1}^m$ for either $\mathbb{C}^n$ or $\mathbb{R}^n$; then

$$\max_{k\neq j} |\langle f_k, f_j\rangle| \geq \sqrt{\frac{m-n}{n(m-1)}}.$$

Equality holds if and only if $\{f_k\}_{k=1}^m$ is an equiangular tight frame.

(i) In the case of $\mathbb{C}^n$, equality can only occur if $m \leq n(n+1)/2$;

(ii) In the case of $\mathbb{R}^n$, equality can only occur if $m \leq n^2$.

Equiangular frames

- The understanding of equiangular tight frames is far from complete;
- The paper by Strohmer & Heath contains examples of equiangular tight frames, e.g., certain versions of the harmonic frames where the columns are generated by different roots of unity.
- More examples of equiangular tight frame and no-go theorems in the papers by Sustik et al., Xia et al., and Strohmer.

Characterization of all dual frames

Result by Shidong Li, 1991:

Theorem: *Let $\{f_k\}_{k=1}^\infty$ be a frame for a Hilbert space $\mathcal{H}$. The dual frames of $\{f_k\}_{k=1}^\infty$ are precisely the families*

$$\{g_k\}_{k=1}^\infty = \left\{ S^{-1}f_k + h_k - \sum_{j=1}^\infty \langle S^{-1}f_k, f_j\rangle h_j \right\}_{k=1}^\infty,$$

where $\{h_k\}_{k=1}^\infty$ is a Bessel sequence in $\mathcal{H}$.

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Allows us to *optimize* the duals:

- Which dual has the best approximation theoretic properties?
- Which dual has the smallest support?
- Which dual has the most convenient expression?
- Can we find a dual that is easy to calculate?

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- Can we find a dual that is easy to calculate?
- Why consider dual frame pairs instead of just tight frames?

An example: Sigma-Delta quantization

Work by Lammers, Powell, and Yilmaz (2009):

Consider a frame $\{f_k\}_{k=1}^m$ for $\mathbb{R}^n$. Letting $\{g_k\}_{k=1}^m$ denote a dual frame, each $f \in \mathbb{R}^n$ can be written

$$f = \sum_{k=1}^{m} \langle f, g_k \rangle f_k.$$

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$$f = \sum_{k=1}^{m} \langle f, g_k \rangle f_k.$$

In practice: the coefficients $\langle f, g_k \rangle$ must be quantized, i.e., replaced by some coefficients d_k from a discrete set such that

$$d_k \approx \langle f, g_k \rangle,$$

which leads to

$$f \approx \sum_{k=1}^{m} d_k f_k.$$

Note: increased redundancy (large m) increases the chance of a good approximation.

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- For each $r \in \mathbb{N}$ there is a procedure (rth order sigma-delta quantization) to find appropriate coefficients d_k.

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- For each $r \in \mathbb{N}$ there is a procedure (rth order sigma-delta quantization) to find appropriate coefficients d_k.
- $r$the order sigma-delta quantization with the canonical dual frame does not provide approximation order m^{-r}, even for tight frames.
- Approximation order m^{-r} can be obtained using other dual frames, the so-called rth order Sobolev duals.

Tight frames versus dual pairs

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Theorem: For each Bessel sequence $\{f_k\}_{k=1}^{\infty}$ in a Hilbert space $\mathcal{H}$, there exists a family of vectors $\{p_j\}_{i\in J}$ such that

$$\{f_k\}_{k=1}^{\infty} \cup \{p_j\}_{i\in J}$$

is a tight frame for $\mathcal{H}$.

Similarly:

Theorem (C., Kim & Kim, 2011) Let $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ be Bessel sequences in $\mathcal{H}$. Then there exist Bessel sequences $\{p_j\}_{i\in J}$ and $\{q_j\}_{i\in J}$ in $\mathcal{H}$ such that $\{f_i\}_{i\in I} \cup \{p_j\}_{i\in J}$ and $\{g_i\}_{i\in I} \cup \{q_j\}_{i\in J}$ form a pair of dual frames for $\mathcal{H}$.

Tight frames versus dual pairs

Example Let $\{e_j\}_{j=1}^{10}$ be an orthonormal basis for $\mathbb{C}^{10}$ and consider the frame

$$\{f_j\}_{j=1}^{10} := \{2e_1\} \cup \{e_j\}_{j=2}^{10}.$$

There exist 9 vectors $\{h_j\}_{j=1}^{9}$ such that

$$\{f_j\}_{j=1}^{10} \cup \{h_j\}_{j=1}^{9}$$

is a tight frame for $\mathbb{C}^{10}$ - and 9 is the minimal number to add.

A pair of dual frames can be obtained by adding just one element:

$$\{f_j\}_{j=1}^{10} \cup \{-3e_1\} \ \text{ and } \ \{f_j\}_{j=1}^{10} \cup \{e_1\}$$

form dual frames in $\mathbb{C}^{10}$.

Tight frames versus dual pairs

Theorem (Casazza and Fickus): Given a sequence of positive numbers $a_1 \geq a_2 \geq \cdots \geq a_m$, there exists a tight frame $\{f_j\}_{j=1}^m$ for $\mathbb{R}^n$ with $||f_j|| = a_j,\ j = 1, \ldots, m$, if and only if

$$a_1^2 \leq \frac{1}{n} \sum_{j=1}^{n} a_j^2. \tag{1}$$

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Theorem (C., Powell, Xiao, 2010): Given any sequence $\{\alpha_j\}_{j=1}^m$ of real numbers, and assume that $m > n$. Then the following are equivalent:

(i) There exist a pair of dual frames $\{f_j\}_{j=1}^m$ and $\{\widetilde{f_j}\}_{j=1}^m$ for $\mathbb{R}^n$ such that $\alpha_j = \langle f_j, \widetilde{f_j} \rangle$ for all $j = 1, \ldots, m$.

(ii) $n = \sum_{j=1}^m \alpha_j$.

Gabor frames - from $L^2(\mathbb{R})$ to $\mathbb{C}^L$

- For $a \in \mathbb{R}$, define the translation operator

$$T_a : L^2(\mathbb{R}) \to L^2(\mathbb{R}), T_a f(x) = f(x-a).$$

- For $b \in \mathbb{R}$, define the modulation operator

$$E_b : L^2(\mathbb{R}) \to L^2(\mathbb{R}), E_b f(x) = e^{2\pi ibx} f(x).$$

- A frame for $L^2(\mathbb{R})$ of the form

$$\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}} = \{e^{2\pi imbx} g(x-na)\}_{m,n\in\mathbb{Z}}$$

is called a Gabor frame.

The duals of a Gabor frame for $L^2(\mathbb{R})$

For a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ with associated frame operator S, the frame decomposition shows that

$$\begin{aligned} f &= \sum_{m,n\in\mathbb{Z}} \langle f, S^{-1}E_{mb}T_{na}g\rangle E_{mb}T_{na}g \qquad [S \text{ commutes with } E_{mb}T_{na}] \\ &= \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}S^{-1}g\rangle E_{mb}T_{na}g, \quad \forall f \in L^2(\mathbb{R}). \end{aligned}$$

Note that the canonical dual of a Gabor frame is again a Gabor frame.

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Note that the canonical dual of a Gabor frame is again a Gabor frame.

But - how can we control the properties of $S^{-1}g$?

Suggestion: Don't construct a nice frame and *expect* the canonical dual to be nice.

The duals of a Gabor frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$

Construct simultaneously dual pairs $\{E_{mb}T_{na}g\},\{E_{mb}T_{na}h\}$ such that g and h have the required properties, and

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \ \forall f \in L^2(\mathbb{R}).$$

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Ron & Shen, A.J.E.M. Janssen (1998):

Theorem: *Two Bessel sequences $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{mb}T_{na}h\}_{m,n\in\mathbb{Z}}$ form dual frames if and only if*

(i) $\sum_{k\in\mathbb{Z}} \overline{g(x-ka)}h(x-ka) = b, \ a.e. \ x \in [0,a].$

(ii) $\sum_{k\in\mathbb{Z}} \overline{g(x-ka-n/b)}h(x-ka) = 0, \ a.e. \ x \in [0,a], \ n \in \mathbb{Z}\setminus\{0\}.$

Explicit construction of dual pairs of Gabor frames in $L^2(\mathbb{R})$

In order for a frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ to be useful, we need a dual frame $\{E_{mb}T_{na}h\}$, i.e., we must find $h \in L^2(\mathbb{R})$ such that

$$f = \sum_{m,n\in\mathbb{Z}} \langle f, E_{mb}T_{na}h\rangle E_{mb}T_{na}g, \ \forall f \in L^2(\mathbb{R}).$$

Ansatz/suggestion: Given a window function $g \in L^2(\mathbb{R})$generating a frame $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$, look for a dual window of the form

$$h(x) = \sum_{k=-K}^{K} c_k g(x+k).$$

The structure of h makes it easy to derive properties of h based on properties of g (regularity, size of support, membership un various vector spaces,....)

Explicit construction of dual pairs of Gabor frames

Theorem:(C., 2006; C. & R. Y. Kim, 2007) *Let* $N \in \mathbb{N}$. *Let* $g \in L^2(\mathbb{R})$ *be a real-valued bounded function for which*

- *supp* $g \subseteq [0, N]$,
- $\sum_{n\in\mathbb{Z}} g(x-n) = 1.$

Let $b \in]0, \frac{1}{2N-1}]$. *Define* $h \in L^2(\mathbb{R})$ *by*

$$h(x) = \sum_{n=-N+1}^{N-1} a_n g(x+n),$$

where

$$a_0 = b, \;\; a_n + a_{-n} = 2b, \; n = 1, 2, \cdots, N-1.$$

Then g *and* h *generate dual frames* $\{E_{mb}T_n g\}_{m,n\in\mathbb{Z}}$ *and* $\{E_{mb}T_n h\}_{m,n\in\mathbb{Z}}$.

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The conditions are satisfied for all B-splines, i.e., the functions B_N where

$$B_1 := \chi_{[0,1]}, \;\; B_{N+1}(x) := B_N * B_1(x) = \int_0^1 B_N(x-t)\, dt.$$

Candidates for g - the B-splines

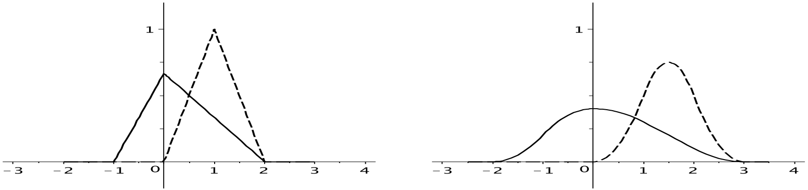
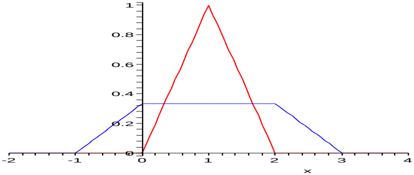
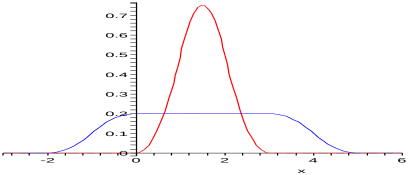


Figure: The B-splines B_2, B_3 and some dual windows





Gabor frames - from $L^2(\mathbb{R})$ to $\mathbb{C}^L$

- Gabor analysis deals with frames $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ for $L^2(\mathbb{R})$.
- For concrete implementations a finite-dimensional model is needed.
- Work initiated by Janssen, 1995: certain Gabor frame for $L^2(\mathbb{R})$ can be transferred into frames for $\ell^2(\mathbb{Z})$ by sampling.
- Søndergaard, Kaiblinger, 2005: certain Gabor frames for $\ell^2(\mathbb{Z})$ can be turned into Gabor frames for $\mathbb{C}^L$ by periodization.

$$\begin{array}{ccc}
L^2(\mathbb{R}) & \xrightarrow{\text{sampling}} & \ell^2(\mathbb{Z}) \\
\Big\downarrow \text{periodization} & & \Big\downarrow \\
L^2(0,L) & \longrightarrow & \mathbb{C}^L
\end{array}$$

Gabor frames - from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z})$

For $g \in \ell^2(\mathbb{Z})$, write the jth coordinate as $g(j)$. Thus,

$$g = (\dots, g(-1), g(0), g(1), \dots).$$

Definition: Gabor systems in $\ell^2(\mathbb{Z})$:

- Given $n \in \mathbb{Z}$ and $g \in \ell^2(\mathbb{Z})$, let $T_n g$ be the sequence in $\ell^2(\mathbb{Z})$ whose jth coordinate is

$$T_n g(j) = g(j-n).$$

- Given $M \in \mathbb{N}$ and $m \in \{0, 1, \dots, M-1\}$, define the action of the modulation operator $E_{m/M}$ on $g \in \ell^2(\mathbb{Z})$ by

$$E_{m/M} g(j) := e^{2\pi imj/M} g(j).$$

- The family of sequences $\{E_{m/M} T_{nN} g\}_{n\in\mathbb{Z}, m=0,\dots,M-1}$ is called the discrete Gabor system generated by the sequence $g \in \ell^2(\mathbb{Z})$ and with modulation parameter $1/M$ and translation parameter N; specifically,

$$E_{m/M} T_{nN} g(j) = e^{2\pi ijm/M} g(j-nN).$$

Gabor frames - from $L^2(\mathbb{R})$ to $\ell^2(\mathbb{Z})$

Given a continuous function $f \in L^2(\mathbb{R})$, define the discrete sequence f^D by

$$f^D := \{f(j)\}_{j\in\mathbb{Z}}.$$

Theorem: Let $M, N \in \mathbb{N}$ be given, and assume that

(i) g and h are two functions, belonging to either $C_c(\mathbb{R})$ or the Feichtinger algebra $\mathcal{S}_0$;

(i) The Gabor systems $\{E_{m/M}T_{nN}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{m/M}T_{nN}h\}_{m,n\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$.

Then the discrete Gabor systems $\{E_{m/M}T_{nN}g^D\}_{n\in\mathbb{Z},m=0,\dots,M-1}$ and $\{E_{m/M}T_{nN}h^D\}_{n\in\mathbb{Z},m=0,\dots,M-1}$ are dual frames for $\ell^2(\mathbb{Z})$; in the case where $g, h \in C_c(\mathbb{R})$, these sequences are finite.

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This applies to all B-splines B_N, $N \geq 2$.

Gabor frames - from $L^2(\mathbb{R})$ to $L^2(0,L)$

Definition: Gabor systems on $L^2(0,L)$: Let $L \in \mathbb{N}$.

- Consider $L^2(0,L)$ as a space of L-periodic functions.
- For $a \in \mathbb{R}$, define the translation operator on $L^2(0,L)$ by

$$T_a : L^2(0,L) \to L^2(0,L),\ T_a f(x) = f(x-a).$$

- The modulation operator on $L^2(0,L)$ is for $b \in L^{-1}\mathbb{Z}$ defined by

$$E_b : L^2(0,L) \to L^2(0,L),\ E_b f(x) = e^{2\pi ibx} f(x).$$

- Fix $L \in \mathbb{N}$, choose $b \in L^{-1}\mathbb{N}$ and $a \in \mathbb{N}$ such that $N := L/a \in \mathbb{N}$. The corresponding *Gabor system* in $L^2(0,L)$ and generated by a function $g \in L^2(0,L)$ is defined by

$$\{E_{mb}T_{na}g\}_{m\in\mathbb{Z},n=0,\ldots,N-1} := \{e^{2\pi ibx}g(x-na)\}_{m\in\mathbb{Z},n=0,\ldots,N-1}.$$

- The periodization operator $\mathcal{P}_L$ on $L^2(\mathbb{R})$ is formally defined by

$$\mathcal{P}_L f(x) := \sum_{k\in\mathbb{Z}} f(x+kL).$$

Gabor frames - from $L^2(\mathbb{R})$ to $L^2(0,L)$

Theorem: Let $\ell, M, N \in \mathbb{N}$. Then the following holds:

(i) If $g \in \mathcal{S}_0$ and $\{E_{m/M}T_{nN}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B, then the periodized Gabor system $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g\}_{n\in\mathbb{Z},m=0,\dots,M\ell-1}$ is a frame for $L^2(0,NM\ell)$ with bounds A, B.

(ii) Let $g, h \in \mathcal{S}_0$. If $\{E_{m/M}T_{nN}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{m/M}T_{nN}h\}_{m,n\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$, then the periodized Gabor systems $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g\}_{n\in\mathbb{Z},m=0,\dots,M\ell-1}$ and $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g\}_{n\in\mathbb{Z},m=0,\dots,M\ell-1}$ are dual frames for $L^2(0,NM\ell)$.

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This applies to all B-splines B_N, $N \geq 2$.

Gabor frames - from $L^2(\mathbb{R})$ to $\mathbb{C}^L$

Definition: Given any $L \in \mathbb{N}$, let $M, N \in \mathbb{N}$ and assume that $M' := L/M \in \mathbb{N}$ and $N' := L/N \in \mathbb{N}$. Given a sequence $g \in \mathbb{C}^L$, define the associated Gabor system on $\mathbb{C}^L$ by

$$\begin{aligned} & \{E_{m/M}T_{nN}g\}_{m=0,\dots,M-1;n=0,\dots,N'-1} \\ = \; & \{e^{2\pi in(\cdot)/M}g(\cdot - nN)\}_{m=0,\dots,M-1;n=0,\dots,N'-1}. \end{aligned}$$

Specifically, $E_{m/M}T_{nN}g$ is the sequence in $\mathbb{C}^L$ whose jth coordinate is

$$E_{m/M}T_{nN}g(j) = e^{2\pi inj/M}g(j - nN).$$

Note that the Gabor system consists of MN' vectors in $\mathbb{C}^L$.

Gabor frames - from $L^2(\mathbb{R})$ to $\mathbb{C}^L$

Theorem Let $N, M, \ell \in \mathbb{N}$ be given. Then the following holds:

(i) If $g \in \mathcal{S}_0$ and the Gabor system $\{E_{m/M}T_{nN}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$ with bounds A, B, then the discrete Gabor system $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g^D\}_{m=0,\dots,M-1,n=0,\dots,M\ell-1}$ is a frame for $\mathbb{C}^{NM\ell}$ with bounds A, B.

(ii) If $g, h \in \mathcal{S}_0$ and the Gabor systems $\{E_{m/M}T_{nN}g\}_{m,n\in\mathbb{Z}}$ and $\{E_{m/M}T_{nN}g\}_{m,n\in\mathbb{Z}}$ are dual frames for $L^2(\mathbb{R})$, then the discrete Gabor systems $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g^D\}_{m=0,\dots,M-1,n=0,\dots,M\ell-1}$ and $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g^D\}_{m=0,\dots,M-1,n=0,\dots,M\ell-1}$ are dual frames for $\mathbb{C}^{NM\ell}$.

$$\begin{tikzcd}
L^2(\mathbb{R}),\ \{E_{m/M}T_{nN}g\} \arrow[r, "\text{sampling}"] \arrow[d, "\text{periodization}"] & \ell^2(\mathbb{Z}),\ \{E_{m/M}T_{nN}g^D\} \arrow[d] \\
L^2(0, NM\ell),\ \{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g\} \arrow[r] & \mathbb{C}^{NM\ell},\ \{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g^D\}
\end{tikzcd}$$

Properties of the finite frame $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g^D\}$

The constructed frame $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g^D\}_{m=0,\dots,M-1,n=0,\dots,M\ell-1}$ for $\mathbb{C}^{NM\ell}$ has several of the attractive properties from the "finite frame wish list:"

- The elements have constant norm;
- The condition number is bounded by the condition number of the given frame $\{E_{m/M}T_{nN}g\}_{m,n\in\mathbb{Z}}$ in $L^2(\mathbb{R})$;
- Explicit versions of the results appear by applications to the B-splines $B_N, N \geq 2$;

Questions related to the finite frame $\{E_{m/M}T_{nN}\mathcal{P}_{NM\ell}g^D\}$

- A Gabor system $\{E_{m/M}T_{nN}g\}_{m=0,\ldots,M-1;n=0,\ldots,N'-1}$ in $\mathbb{C}^L$ is known to have full spark for a.e. $g \in \mathbb{C}^L$ (proved for L prime by Lawrence, Pfander, and Walnut (2005), and in full generality by Malikiosis (2013).

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- Question: Do the Gabor systems $\{E_{m/M}T_{nN}g\}_{m=0,\ldots,M-1;n=0,\ldots,N'-1}$ constructed via sampling and periodization have full spark? E.g., if the B-splines $B_N, N \geq 2$, are used as windows?

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- Equiangular tight Gabor frames are considered by Fickus (2009).

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- Equiangular tight Gabor frames are considered by Fickus (2009).
- Question: Are (some of) the Gabor systems $\{E_{m/M}T_{nN}g\}_{m=0,\ldots,M-1;n=0,\ldots,N'-1}$ constructed via sampling and periodization equiangular? E.g., if the B-splines $B_N, N \geq 2$, are used as windows?

Final remarks

- The similarity between the definitions and properties of the Gabor systems on $L^2(\mathbb{R}), \ell^2(\mathbb{Z}), L^2(0,L)$, and $\mathbb{C}^L$ is not a coincidence: the sets $\mathbb{R}, \mathbb{Z}, [0,L[$ and $\mathbb{Z}_L$ can all be regarded as locally compact abelian groups, and the general theory for Gabor systems on LCA groups applies.
- Letting $\ell \to \infty$ yields Gabor systems in high-dimensional sequence spaces and a method for approximation of the inverse frame operator.
- Søndergaard has implementet the LTFAT Matlab toolbox, which allows to perform finite-dimensional frame calculations (e.g., computation of the dual frame).

An alternative way to obtain finite "Gabor systems"

Theorem Suppose that $ab < 1$ and that $\{E_{mb}T_{na}g\}_{m,n\in\mathbb{Z}}$ is a frame for $L^2(\mathbb{R})$. For $N \in \mathbb{N}$, let $\mathcal{E}_N$ denote a lower frame bound for the frame sequence $\{E_{mb}T_{na}g\}_{|m|,|n|\leq N}$. Then

$$\mathcal{E}_N \to 0 \text{ as } N \to \infty.$$

Thus, the "cut-off" procedure is not suitable for obtaining well-conditioned finite-dimensional systems!

A conjecture by Heil, Ramanathan, and Topiwala (1995)

The HRT-Conjecture: Given any finite collection of distinct points $\{(\mu_k, \lambda_k)\}_{k\in\mathcal{F}}$ in $\mathbb{R}^2$ and a function $g \neq 0$, the Gabor system

$$\{e^{2\pi i\lambda_k x} g(x-\mu_k)\}_{k\in\mathcal{F}}$$

is linearly independent.

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is linearly independent.

The conjecture has been confirmed for regular Gabor frames $\{E_{mb} T_{na} g\}_{m,n \in \mathbb{Z}}$ and some irregular Gabor systems, but the general case is still open.

Dedicated to John Benedetto and Hans Feichtinger

- Alexeev, B., C., and Mixon, D. G.: *Full spark frames.* J. Fourier Anal. Appl. **18** no. 6 (2012), 1167–1194.
- Benedetto, J., and Fickus, M.: *Finite normalized tight frames.* Adv. Comp. Math. **18** (2003), no. 2-4, 357–385.
- Benedetto, J., Powell A., and Yilmaz, Ö.: *Sigma-Delta quantization and finite frames. IEEE. Trans. Inform. Theory*, **52** (2006), 1990–2005.
- Blum, J., Lammers, M., Powell, A. M., and Yilmaz, Ö.: *Sobolev duals in frame theory and sigma-delta quantization.* J. Fourier Anal. Appl. **16** (2010), 365–381.
- Bodmann, B. G., Casazza, P. G.; Kutyniok, G.: *A quantitative notion of redundancy for finite frames.* Appl. Comp. Harm. Anal. **30** (2011), no. 3, 348–362.
- Casazza, P.G. and Kovačević, J.: *Equal-norm tight frames with erasures.* Adv. Comput. Math. **18** (2003), 387–430.

- Casazza, P. G. and Kutyniok, G. (eds): *Finite frames. Theory and applications.* Birkhäuser 2012.
- Casazza, P.G. and Leon, M.: *Existence and construction of finite tight frames.* J. Concr. Appl. Math. **4** (2006), no. 3, 277–289.
- Casazza, P. G., and Leonhard, N.: *Classes of finite equal norm Parseval frames*, Contemp. Math. **451** (2008), 11–31.
- Christensen, O.: *Pairs of dual Gabor frames with compact support and desired frequency localization.* Appl. Comp. Harm. Anal. **20** (2006), 403–410.
- Christensen, O. and Kim, R. Y.: *Pairs of explicitly given dual Gabor frames in* $L^2(R^d)$. J. Fourier Anal. Appl. **12** vol. 3 (2006), 243–255.
- Christensen, O., Powell, A. M. and Xiao, X. C.: *A note on finite dual frame pairs.* Proc. Amer. Math. Soc. **140** no. 11 (2012), 3921–3930.
- Fickus, M.: *Maximally equiangular frames and Gauss sums.* J. Fourier Anal. Appl. **15** no. 3 (2009), 413–427.

Janssen, A.J.E.M.: *From continuous to discrete Weyl-Heisenberg frames through sampling.* J. Fourier Anal. Appl. **3** no. 5 (1997), 583–596.

Kaiblinger, N.: *Approximation of the Fourier transform and the dual Gabor window.* J. Fourier Anal. Appl. **11** no.1 (2005), 25–42.

Lawrence, J., Pfander, G. E., and Walnut, D.: *Linear independence of Gabor systems in finite dimensional vector spaces.* J. Fourier Anal. Appl. **11** no. 6 (2005), 715–726.

Li, S.: *On general frame decompositions*. Numer. Funct. Anal. and Optimiz. **16** no. 9 & 10 (1995), 1181–1191.

Malikiosis, R.-D.: *A note on Gabor frames in finite dimensions.* Appl. Comp. Harm. Anal. **38** (2015), 318–330.

Strohmer, T.: *A note on equiangular tight frames.* Lin. Alg. Appl. **429** no. 1 (2008), 326–330.

- Strohmer, T. and Heath, R.: *Grassmannian frames with applications to coding and communication.* Appl. Comp. Harmon. Anal. **14** (2003), 257-275.
- Sustik, M. A., Tropp, J. A., Dhillon, I. S., and Heath, R. W., Jr.: *On the existence of equiangular tight frames.* Lin. Alg. Appl. **426** no. 2-3 (2007), 619–635.
- Søndergaard, P.: *Gabor frames by sampling and periodization.* Adv. Comput. Math. **27** (2007), no. 4, 355–373.
- Søndergaard, P., Torresani, B., and Balazs, P.: *The linear time frequency analysis toolbox.* Int. J. Wavelets Multiresolut. Inf. Process. **10** (2012), no. 4.
- Xia, P., Zhou, S., and Giannakis, S. B.: *Achieving the Welch Bound with difference sets.* IEEE Trans. Inform. Theory **51** no. 5 (2005), 1900–1907.
- Zimmermann, G.: *Normalized tight frames in finite dimensions.* In "Recent progress in multivariate approximation", 249–252. (eds. Jetter, K., Haußmann, W., Reimer, M.). Birkhäuser, Boston, 2001.